

# IMPROVED SUBCONVEXITY BOUNDS FOR $GL(2) \times GL(3)$ AND $GL(3)$ $L$ -FUNCTIONS BY WEIGHTED STATIONARY PHASE

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**ABSTRACT.** Let  $f$  be a fixed self-contragradient Hecke-Maass form for  $SL(3, \mathbb{Z})$ , and  $u$  an even Hecke-Maass form for  $SL(2, \mathbb{Z})$  with Laplace eigenvalue  $1/4 + k^2$ ,  $k > 0$ . A subconvexity bound  $O(k^{4/3+\varepsilon})$  in the eigenvalue aspect is proved for the central value at  $s = 1/2$  of the Rankin-Selberg  $L$ -function  $L(s, f \times u)$ . Meanwhile, a subconvexity bound  $O((1+|t|)^{2/3+\varepsilon})$  in the  $t$  aspect is proved for  $L(1/2+it, f)$ . These bounds improved corresponding subconvexity bounds proved by Xiaoqing Li (Annals of Mathematics, 2011). The main technique in the proof, other than those used by Li, is an  $n$ th-order asymptotic expansion of a weighted stationary phase integral, for arbitrary  $n \geq 1$ . This asymptotic expansion sharpened the classical result for  $n = 1$  by Huxley.

## 1. INTRODUCTION

Bounds for automorphic  $L$ -functions on the critical line  $\text{Re}(s) = 1/2$  are central questions in number theory and have far-reaching applications (cf. Iwaniec and Sarnak [12] and Michel [22]). The ultimate conjectured bounds are predicted by the Lindelöf Hypothesis, while trivial bounds include the convexity bounds as a consequence of the Phragmén-Lindelöf principle. Any bound which have a power saving over the corresponding convexity bound is highly non-trivial and called a subconvexity bound.

The strength of a subconvexity bound is crucial. There are important applications which depend on the strength of the subconvexity bounds. A notable example is the number of real zeros of a holomorphic Hecke cusp form  $f$  for  $SL(2, \mathbb{Z})$  of weight  $k$ , i.e., zeros of  $f$  on  $\{iy|y \geq 1\}$ . By Ghosh and Sarnak [5], the number of such zeros is  $\gg \log k$ . Their proof uses a Weyl-like, i.e., a  $1/3$  power-saving, subconvexity bound for  $L(s, f)$  proved by Peng [24] and Jutila and Motohashi [14]. Note that a subconvexity bound for  $L(s, f)$  with a power saving less than  $1/3$  does not suffice in [5].

In this paper, we will prove subconvexity bounds for certain Rankin-Selberg  $L$ -functions for  $GL(3) \times GL(2)$  and automorphic  $L$ -functions for  $GL(3)$  over  $\mathbb{Q}$  which improve bounds established by Xiaoqing Li [19].

**Theorem 1.1.** *Let  $f$  be a fixed self-contragradient Hecke-Maass form for  $SL(3, \mathbb{Z})$  normalized by  $A(1, 1) = 1$ , and  $\{u_j\}$  an orthonormal basis of even Hecke-Maass forms for  $SL(2, \mathbb{Z})$ . Denote by  $1/4 + t_j^2$ ,  $t_j \geq 0$ , the Laplace eigenvalue of  $u_j$ . Then for large  $T$  and  $T^{1/3+\varepsilon} \leq M \leq T^{1/2}$  we have*

$$(1.1) \quad \sum_j e^{-(t_j-T)^2/M^2} L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-(t-T)^2/M^2} \left| L\left(\frac{1}{2} - it, f\right) \right|^2 dt \ll_{\varepsilon, f} T^{1+\varepsilon} M$$

for any  $\varepsilon > 0$ .

Note that in [19] the same (1.1) was proved for  $T^{3/8+\varepsilon} \leq M \leq T^{1/2}$ . As pointed out in [19],

$$(1.2) \quad L\left(\frac{1}{2}, f \times u_j\right) \geq 0$$

was proved by Lapid [16] because  $f$  is orthogonal and  $u_j$  is symplectic (Jacquet and Shalika [13]).

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We remark that the normalization of  $u_j$  is different from the normalization  $\lambda_{u_j}(1) = 1$  as required in the definition of  $L(s, f \times u_j)$ , but the discrepancy is within  $t_j^\varepsilon$  as proved in Hoffstein and Lockhart [9].

The smaller allowable power of  $T$  for  $M$  in Theorem 1.1 gives us a smaller subconvexity bound.

**Corollary 1.2.** *Let  $f$  be a fixed self-contragradient Hecke-Maass form for  $SL(3, \mathbb{Z})$  normalized by  $A(1, 1) = 1$ , and  $u$  an even Hecke-Maass form for  $SL(2, \mathbb{Z})$  normalized by  $\lambda_u(1) = 1$ . Denote by  $1/4 + k^2$ ,  $k > 0$ , the Laplace eigenvalue of  $u$ .*

$$L\left(\frac{1}{2}, f \times u\right) \ll_{\varepsilon, f} k^{4/3+\varepsilon}.$$

Note that Corollary 1.2 improved the bound  $O(k^{11/8+\varepsilon})$  proved in [19]. The convexity bound is  $O(k^{3/2+\varepsilon})$ .

Because of the nonnegativity (1.2), the bound in (1.1) implies a square moment bound for  $L(s, f)$  over a short interval.

**Corollary 1.3.** *Let  $f$  be a fixed self-contragradient Hecke-Maass form for  $SL(3, \mathbb{Z})$  normalized by  $A(1, 1) = 1$ . Then for  $T^{1/3+\varepsilon} \leq M \leq T^{1/2}$*

$$(1.3) \quad \int_{-\infty}^{\infty} e^{-(t-T)^2/M^2} \left| L\left(\frac{1}{2} - it, f\right) \right|^2 dt \ll_{\varepsilon, f} T^{1+\varepsilon} M.$$

Since  $f$  is a  $GL(3)$  form, the square moment in (1.3) is comparable to a sixth power moment of the Riemann zeta function. Similar arguments were carried out for a  $GL(2)$  form in Ye [28] and Lau, Liu and Ye [17].

By a standard argument of analytic number theory (cf. Heath-Brown [8] or Ivić [11], p. 197), we derived a subconvexity bound for  $L(s, f)$  in the  $t$  aspect. Its improvement over [19]'s  $O((1 + |t|)^{11/16+\varepsilon})$  is again based on the smaller allowable power of  $T$  for  $M$ . The convexity bound is  $O((1 + |t|)^{3/4+\varepsilon})$ .

**Corollary 1.4.** *Let  $f$  be a fixed self-contragradient Hecke-Maass form for  $SL(3, \mathbb{Z})$  normalized by  $A(1, 1) = 1$ . Then*

$$L\left(\frac{1}{2} + it, f\right) \ll_{\varepsilon, f} (1 + |t|)^{2/3+\varepsilon}.$$

Following Ye and Deyu Zhang [29], we can deduce the following result on zero density for  $L(s, f)$  from (1.3). Let

$$N_f(\sigma, T, T + T^\delta) = \#\{\rho = \beta + i\gamma \mid L(\rho, f) = 0, \sigma < \beta < 1, T \leq \gamma \leq T + T^\delta\}$$

be the number of zeros of  $L(s, f)$  in the box of  $\sigma < \beta < 1$  and  $T \leq \gamma \leq T + T^\delta$ .

**Corollary 1.5.** *Let  $f$  be a fixed self-contragradient Hecke-Maass form for  $SL(3, \mathbb{Z})$ . Then for  $1/3 < \delta \leq 1$ , we have*

$$(1.4) \quad \begin{aligned} N_f(\sigma, T, T + T^\delta) &\ll_{\varepsilon, f} T^{\frac{(2+4\delta)(1-\sigma)}{3-2\sigma} + \varepsilon} \quad \text{for } 1/2 \leq \sigma < \frac{2+\delta}{2+2\delta}; \\ &\ll_{\varepsilon, f} T^{2(1+\delta)(1-\sigma) + \varepsilon} \quad \text{for } \frac{2+\delta}{2+2\delta} \leq \sigma < 1. \end{aligned}$$

We remark that Corollary 1.5 shows that (1.4) is now valid on a shorter interval  $[T, T + T^\delta]$  with  $1/3 < \delta \leq 1$  than the interval with  $3/8 < \delta \leq 1$  in [29] which uses Li [19].

As noted in [19], Theorem 1.1 can also be proved for  $f$  being the minimal Eisenstein series on  $GL(3)$ . This has been carried out in Lu [21]. Our proof and improvement can also be applied to that case.

The main techniques of our proof, other than those used in [19], are asymptotic expansions of exponential integrals

$$(1.5) \quad \int_{\alpha}^{\beta} g(x) e(f(x)) dx$$

when  $f'(x)$  changes signs at a point  $x = \gamma$  with  $\alpha < \gamma < \beta$ . Huxley [10] obtained the first-order asymptotic expansion of (1.5). His results [10] are used widely as standard techniques in analytic number theory and other branches of mathematics.

What we need in our proof, however, is an asymptotic expansion of (1.5) beyond the first order. Blomer, Khan and Young [3] got such an asymptotic expansion, but for the case in hand we need more explicit main terms (Theorem 3.6):

$$\begin{aligned} \int_{\alpha}^{\beta} g(x) e(f(x)) dx &= \frac{e(f(\gamma) + 1/8)}{\sqrt{|f''(\gamma)|}} \left( g(\gamma) + \sum_{j=1}^n \varpi_{2j} \frac{(-1)^j (2j-1)!!}{(2\pi i f''(\gamma))^j} \right) \\ &\quad + \text{Boundary terms} + \text{Error terms.} \end{aligned}$$

Here  $n$  is related to how many derivatives we assume  $f$  and  $g$  have. The functions  $\varpi_{2j}$  (3.38) are related to the Taylor approximation of  $g$  centered at  $\gamma$ , as well as changing the measure  $dx$  to  $dy$  through the transformation  $f''(\gamma) \cdot y^2 = 2(f(x) - f(\gamma))$ . See (3.43) and (3.45) for the case of  $n = 2$ . We will apply Voronoi's formula to the leading term of  $\varpi_{2j}$  the second time and estimate its other terms trivially.

In the following sections,  $\varepsilon$  is any arbitrarily small positive number. Its value may be different on each occurrence.

## 2. THE FIRST DERIVATIVE TESTS

The following theorem is the weighted first derivative test, which strengthens Lemma 5.5.5 of [10], p.113, with more boundary terms and smaller error terms.

**Theorem 2.1.** *Let  $f(x)$  be a real function,  $n+2$  times continuously differentiable for  $\alpha \leq x \leq \beta$ , and let  $g(x)$  be a real function,  $n+1$  times continuously differentiable for  $\alpha \leq x \leq \beta$ . Suppose that there are positive parameters  $M, N, T, U$ , with  $M \geq \beta - \alpha$ , and positive constants  $C_r$  such that for  $\alpha \leq x \leq \beta$ ,*

$$|f^{(r)}(x)| \leq C_r \frac{T}{M^r}, \quad |g^{(s)}(x)| \leq C_s \frac{U}{N^s},$$

for  $r = 2, \dots, n+2$ , and  $s = 0, \dots, n+1$ . If  $f'(x)$  and  $f''(x)$  do not change signs on the interval  $[\alpha, \beta]$ , then we have

$$(2.1) \quad \begin{aligned} \int_{\alpha}^{\beta} g(x) e(f(x)) dx &= \left[ e(f(x)) \sum_{i=1}^n H_i(x) \right]_{\alpha}^{\beta} \\ &\quad + O\left( \frac{M}{N} \sum_{j=1}^{[n/2]} \frac{UT^j}{\min |f'|^{n+j+1} M^{2j}} \sum_{t=j}^{n-j} \frac{1}{N^{n-j-t} M^t} \right) \end{aligned}$$

$$(2.2) \quad + O\left( \left( \frac{M}{N} + 1 \right) \frac{U}{N^n \min |f'|^{n+1}} \right)$$

$$(2.3) \quad + O\left( \sum_{j=1}^n \frac{UT^j}{\min |f'|^{n+j+1} M^{2j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t} M^t} \right),$$

where

$$(2.4) \quad H_1(x) = \frac{g(x)}{2\pi i f'(x)}, \quad H_i(x) = -\frac{H'_{i-1}(x)}{2\pi i f'(x)}$$

for  $i = 2, \dots, n$ .

If  $g(x) \equiv 1$  on  $[\alpha, \beta]$ , we may take  $U = 1$  and  $N$  arbitrarily large. Then the error terms (2.1) and (2.2) in Theorem 2.1 are negligible, while in the error term (2.3) we may take only one term with  $t = n - j$  in the inner sum. This way we get the explicit first derivative test, which supersedes Lemma 5.5.1 of [10], p.104, with more boundary terms and smaller error terms.

**Theorem 2.2.** *Let  $f(x)$  be a real function,  $n + 2$  times continuously differentiable for  $\alpha \leq x \leq \beta$ . Suppose that there are positive parameters  $M, T$ , with  $M \geq \beta - \alpha$ , and positive constants  $C_r$  such that for  $\alpha \leq x \leq \beta$ ,*

$$|f^{(r)}(x)| \leq C_r \frac{T}{M^r},$$

for  $r = 2, 3, \dots, n + 2$ . If  $f'(x)$  and  $f''(x)$  do not change sign on the interval  $[\alpha, \beta]$ , then we have

$$\int_{\alpha}^{\beta} e(f(x)) dx = \left[ e(f(x)) \sum_{k=1}^n H_k(x) \right]_{\alpha}^{\beta} + O\left( \sum_{t=0}^{n-1} \frac{T^{t+1}}{\min |f'|^{n+2+t} M^{n+1+t}} \right),$$

where

$$H_1(x) = \frac{1}{2\pi i f'(x)}, \quad H_k(x) = -\frac{H'_{k-1}(x)}{2\pi i f'(x)} \text{ for } k = 2, \dots, n.$$

*Proof of Theorem 2.1.* By integration by parts we get

$$(2.5) \quad \int_{\alpha}^{\beta} g(x) e(f(x)) dx = \left[ e(f(x)) \sum_{i=1}^n H_i(x) \right]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} e(f(x)) H'_n(x) dx,$$

where  $H_i(x)$  for  $i = 1, \dots, n + 2$  are given in (2.4). Now we prove

$$(2.6) \quad \begin{aligned} H_k(x) &= \frac{C_{k,0} g^{(k-1)}}{(f')^k} \\ &+ \sum_{j=1}^{k-1} \frac{1}{(f')^{k+j}} \sum_{t=0}^{k-1-j} C_{j,t} g^{(k-1-j-t)} \sum_{\substack{2 \leq n_1, \dots, n_j \leq t+2 \\ n_1 + \dots + n_j = t+2j}} f^{(n_1)} \dots f^{(n_j)} \end{aligned}$$

by induction. By (2.4)

$$H_2(x) = -\frac{1}{2\pi i f'(x)} \left( \frac{g'(x)}{2\pi i f'(x)} - \frac{g(x) f''(x)}{2\pi i (f'(x))^2} \right)$$

which is (2.6) for  $k = 2$ . Suppose (2.6) holds for  $k$ . We will differentiate (2.6) and then divide it by  $-2\pi i f'(x)$ . To simplify the notation, we will ignore all constant coefficients in computation:

$$\begin{aligned}
(2.7) \quad \frac{H'_k(x)}{f'(x)} &= \frac{g^{(k)}}{(f')^{k+1}} \\
(2.8) \quad &+ \frac{g^{(k-1)} f''}{(f')^{k+2}} \\
(2.9) \quad &+ \sum_{j=1}^{k-1} \frac{f''}{(f')^{k+2+j}} \sum_{t=0}^{k-1-j} g^{(k-1-j-t)} \sum_{\substack{2 \leq n_1, \dots, n_j \leq t+2 \\ n_1 + \dots + n_j = t+2j}} f^{(n_1)} \dots f^{(n_j)} \\
(2.10) \quad &+ \sum_{j=1}^{k-1} \frac{1}{(f')^{k+1+j}} \sum_{t=0}^{k-1-j} g^{(k-j-t)} \sum_{\substack{2 \leq n_1, \dots, n_j \leq t+2 \\ n_1 + \dots + n_j = t+2j}} f^{(n_1)} \dots f^{(n_j)} \\
(2.11) \quad &+ \sum_{j=1}^{k-1} \frac{1}{(f')^{k+1+j}} \sum_{t=0}^{k-1-j} g^{(k-1-j-t)} \sum_{\substack{2 \leq n_1, \dots, n_j \leq t+2 \\ n_1 + \dots + n_j = t+2j}} \\
&\quad \times \left( f^{(n_1+1)} f^{(n_2)} \dots f^{(n_j)} + \dots + f^{(n_1)} \dots f^{(n_{j-1}+1)} f^{(n_j+1)} \right).
\end{aligned}$$

Note that (2.7) is the first term in

$$(2.12) \quad H_{k+1}(x) = \frac{g^{(k)}}{(f')^{k+1}} + \sum_{j=1}^k \frac{1}{(f')^{k+1+j}} \sum_{t=0}^{k-j} g^{(k-j-t)} \sum_{\substack{2 \leq n_1, \dots, n_j \leq t+2 \\ n_1 + \dots + n_j = t+2j}} f^{(n_1)} \dots f^{(n_j)}$$

which is what we want to prove. Then (2.8) is the term in (2.12) with  $j = 1$  and  $t = 0$ ; (2.9) is the terms in (2.12) with  $2 \leq j \leq k$  after we change  $j+1$  in (2.9) to  $j$  in (2.12); (2.10) is also contained in (2.12) with the same  $j$  and  $t$ ; and finally (2.11) is the terms in (2.12) with the same  $j$  but for  $1 \leq t \leq k-j$  after we change  $t+1$  in (2.11) to  $t$  in (2.12). Since (2.7)–(2.11) all fit into (2.12) with appropriate coefficients, (2.6) holds for  $k+1$ .

Therefore for  $2 \leq k \leq n+2$  we have from (2.6)

$$\begin{aligned}
(2.13) \quad H_k(x) &\ll \frac{U}{N^{k-1} |f'(x)|^k} + \sum_{j=1}^{k-1} \frac{UT^j}{|f'(x)|^{k+j} M^{2j}} \sum_{t=0}^{k-1-j} \frac{1}{N^{k-1-j-t} M^t} \\
&\ll \frac{U}{N^{k-1} \min |f'|^k} + \sum_{j=1}^{k-1} \frac{UT^j}{\min |f'|^{k+j} M^{2j}} \sum_{t=0}^{k-1-j} \frac{1}{N^{k-1-j-t} M^t}.
\end{aligned}$$

By (2.13) we have

$$\begin{aligned}
(2.14) \quad H'_n(x) &= -2\pi i f'(x) H_{n+1}(x) \\
&\ll \frac{U}{N^n \min |f'|^n} + \sum_{j=1}^n \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t} M^t}.
\end{aligned}$$

Let  $V(\cdot)$  be the total variation. We have

$$\begin{aligned}
V(H'_n(x)) &= \int_{\alpha}^{\beta} |H''_n(x)| dx \\
&= \int_{\alpha}^{\beta} \left| -2\pi i f''(x) H_{n+1}(x) - 2\pi i f'(x) H'_{n+1}(x) \right| dx.
\end{aligned}$$

Note that  $f''(x)$  does not change signs on the interval  $[\alpha, \beta]$ . Hence we have

$$(2.15) \quad \int_{\alpha}^{\beta} \frac{|f''(x)|}{(f'(x))^2} dx = \left| \int_{\alpha}^{\beta} \frac{f''(x)}{(f'(x))^2} dx \right| = \left| \frac{1}{f'(\alpha)} - \frac{1}{f'(\beta)} \right| \leq \frac{2}{\min |f'|}.$$

Using (2.15) and (2.13) we get

$$(2.16) \quad \begin{aligned} & \int_{\alpha}^{\beta} |f''(x) H_{n+1}(x)| dx \\ & \ll \left[ \frac{U}{N^n \min |f'|^{n-1}} + \sum_{j=1}^n \frac{UT^j}{\min |f'|^{n-1+j} M^{2j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t} M^t} \right] \int_{\alpha}^{\beta} \frac{|f''(x)|}{(f'(x))^2} dx \\ & \ll \frac{U}{N^n \min |f'|^n} + \sum_{j=1}^n \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t} M^t}. \end{aligned}$$

By (2.6) we see that the derivative of  $H'_{n+1}(x)$  can be divided into two sums. One sum comes from the derivative of the denominator, which brings in an  $f''$  and is thus called the  $f''$  sum. That is,

$$-2\pi i f'(x) H'_{n+1}(x) = (\text{the } f'' \text{ sum}) + (\text{the other term}).$$

By (2.6) we see that the  $f''$  sum is (ignoring all coefficients)

$$(2.17) \quad f'' \left[ \frac{g^{(n)}}{(f')^{n+1}} + \sum_{j=1}^n \frac{1}{(f')^{n+1+j}} \sum_{t=0}^{n-j} g^{(n-j-t)} \sum_{\substack{2 \leq n_1, \dots, n_j \leq t+2 \\ n_1 + \dots + n_j = t+2j}} f^{(n_1)} \dots f^{(n_j)} \right].$$

The other sum equals

$$(2.18) \quad \begin{aligned} & \frac{g^{(n+1)}}{(f')^n} + \sum_{j=1}^n \frac{1}{(f')^{n+j}} \sum_{t=0}^{n-j} \left[ g^{(n-j-t)} \sum_{\substack{2 \leq n_1, \dots, n_j \leq t+2 \\ n_1 + \dots + n_j = t+2j}} f^{(n_1)} \dots f^{(n_j)} \right]' \\ & = \frac{g^{(n+1)}}{(f')^n} + \sum_{j=1}^n \frac{1}{(f')^{n+j}} \sum_{t=0}^{n-j} g^{(n-j-t+1)} \sum_{\substack{2 \leq n_1, \dots, n_j \leq t+2 \\ n_1 + \dots + n_j = t+2j}} f^{(n_1)} \dots f^{(n_j)} \\ & \quad + \sum_{j=1}^n \frac{1}{(f')^{n+j}} \sum_{t=0}^{n-j} g^{(n-j-t)} \sum_{\substack{2 \leq n_1, \dots, n_j \leq t+2 \\ n_1 + \dots + n_j = t+2j}} \left[ f^{(n_1)} \dots f^{(n_j)} \right]' \\ & =: \frac{g^{(n+1)}}{(f')^n} + \sum_1 + \sum_2. \end{aligned}$$

Now we consider the integral

$$\int_{\alpha}^{\beta} \left| -2\pi i f'(x) H'_{n+1}(x) \right| dx.$$

By (2.15) we see that the contribution of (2.17) is

$$(2.19) \quad \begin{aligned} & \ll \left[ \frac{U}{N^n \min |f'|^{n-1}} + \sum_{j=1}^n \frac{UT^j}{\min |f'|^{n-1+j} M^{2j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t} M^t} \right] \int_{\alpha}^{\beta} \frac{|f''(x)|}{(f'(x))^2} dx \\ & \ll \frac{U}{N^n \min |f'|^n} + \sum_{j=1}^n \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t} M^t}, \end{aligned}$$

while the contribution of  $g^{(n+1)}/(f')^n$  and  $\sum_2$  in (2.18) is

$$(2.20) \quad \begin{aligned} &\ll \left[ \frac{U}{N^{n+1} \min |f'|^n} + \sum_{j=1}^n \frac{U}{\min |f'|^{n+j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t}} \frac{T^j}{M^{t+2j+1}} \right] (\beta - \alpha) \\ &\ll \frac{UM}{N^{n+1} \min |f'|^n} + \sum_{j=1}^n \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t} M^t}. \end{aligned}$$

For  $\sum_1$  in (2.18), we see that if  $t < j$  then there is at least one  $f''(x)$  in the sum of  $f^{(n_1)} \dots f^{(n_j)}$ , since in this case  $n_1 + \dots + n_j < 3j$  and  $n_i \geq 2$ . Note that if  $j > n/2$  then  $t \leq n - j < j$ . Therefore

$$\begin{aligned} \int_{\alpha}^{\beta} |\sum_1| dx &\ll \sum_{j=1}^{[n/2]} \frac{1}{\min |f'|^{n+j}} \sum_{t=j}^{n-j} \frac{U}{N^{n-j-t+1}} \frac{T^j}{M^{t+2j}} (\beta - \alpha) \\ &\quad + \sum_{j=1}^{[n/2]} \frac{1}{\min |f'|^{n+j-2}} \sum_{t=0}^{j-1} \frac{U}{N^{n-j-t+1}} \frac{T^{j-1}}{M^{t+2j-2}} \int_{\alpha}^{\beta} \frac{|f''(x)|}{(f'(x))^2} dx \\ &\quad + \sum_{j>n/2}^n \frac{1}{\min |f'|^{n+j-2}} \sum_{t=0}^{n-j} \frac{U}{N^{n-j-t+1}} \frac{T^{j-1}}{M^{t+2j-2}} \int_{\alpha}^{\beta} \frac{|f''(x)|}{(f'(x))^2} dx, \end{aligned}$$

because in the last two cases there must be an  $f''(x)$ . By (2.15)

$$\begin{aligned} \int_{\alpha}^{\beta} |\sum_1| dx &\ll \frac{M}{N} \sum_{j=1}^{[n/2]} \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=j}^{n-j} \frac{1}{N^{n-j-t} M^t} \\ &\quad + \sum_{j=1}^{[n/2]} \frac{UT^{j-1}}{\min |f'|^{n+j-1} M^{2(j-1)}} \sum_{t=0}^{j-1} \frac{1}{N^{n-j-t+1} M^t} \\ &\quad + \sum_{j>n/2}^n \frac{UT^{j-1}}{\min |f'|^{n+j-1} M^{2(j-1)}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t+1} M^t}. \end{aligned}$$

Now we change  $j \rightarrow j+1$  in the last two sums to get

$$(2.21) \quad \begin{aligned} \int_{\alpha}^{\beta} |\sum_1| dx &\ll \frac{M}{N} \sum_{j=1}^{[n/2]} \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=j}^{n-j} \frac{1}{N^{n-j-t} M^t} \\ &\quad + \sum_{j=0}^{[n/2]-1} \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=0}^j \frac{1}{N^{n-j-t} M^t} \\ &\quad + \sum_{j>n/2-1}^{n-1} \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=0}^{n-j-1} \frac{1}{N^{n-j-t} M^t}. \end{aligned}$$

Obviously the last two terms in (2.21) are contained in (2.19) which is an upper bound for the  $f''$  sum.

Therefore by (2.19), (2.20) and (2.21) we get

$$(2.22) \quad \begin{aligned} &\int_{\alpha}^{\beta} \left| -2\pi i f'(x) H'_{n+1}(x) \right| dx \\ &\ll \left( \frac{M}{N} + 1 \right) \frac{U}{N^n \min |f'|^n} + \sum_{j=1}^n \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t} M^t} \\ &\quad + \frac{M}{N} \sum_{j=1}^{[n/2]} \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=j}^{n-j} \frac{1}{N^{n-j-t} M^t}. \end{aligned}$$

Now from (2.16) and (2.22) we get

$$\begin{aligned}
 (2.23) \quad V(H'_n(x)) &\ll \left(\frac{M}{N} + 1\right) \frac{U}{N^n \min |f'|^n} \\
 &+ \sum_{j=1}^n \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t} M^t} \\
 &+ \frac{M}{N} \sum_{j=1}^{[n/2]} \frac{UT^j}{\min |f'|^{n+j} M^{2j}} \sum_{t=j}^{n-j} \frac{1}{N^{n-j-t} M^t}.
 \end{aligned}$$

At last combining (2.14) and (2.23), by the first derivative test (Lemma 5.1.2 of [10], p.88), we get

$$\begin{aligned}
 \int_{\alpha}^{\beta} e(f(x)) H'_n(x) dx &\ll \frac{V(H'_n(x)) + \max\{|H'_n(x)|\}}{\min |f'(x)|} \\
 &\ll \left(\frac{M}{N} + 1\right) \frac{U}{N^n \min |f'(x)|^{n+1}} \\
 &+ \sum_{j=1}^n \frac{UT^j}{\min |f'(x)|^{n+j+1} M^{2j}} \sum_{t=0}^{n-j} \frac{1}{N^{n-j-t} M^t} \\
 &+ \frac{M}{N} \sum_{j=1}^{[n/2]} \frac{UT^j}{\min |f'(x)|^{n+j+1} M^{2j}} \sum_{t=j}^{n-j} \frac{1}{N^{n-j-t} M^t}.
 \end{aligned}$$

Then by (2.5) we proved the theorem.  $\square$

### 3. THE WEIGHTED STATIONARY PHASE INTEGRAL

Let  $f(x)$  be a real function,  $2n+3$  times continuously differentiable for  $\alpha \leq x \leq \beta$ . Suppose that  $f'(x)$  changes signs only at  $x = \gamma$ , from negative to positive, with  $\alpha < \gamma < \beta$ . We also assume that (3.54) and (3.55) hold. Then the Taylor expansions about  $x = \gamma$  give

$$(3.1) \quad f(x) = f(\gamma) + \sum_{k=2}^{2n+2} \lambda_k (x - \gamma)^k + O_n\left(\frac{T|x - \gamma|^{2n+3}}{M^{2n+3}}\right),$$

$$(3.2) \quad f'(x) = \sum_{k=2}^{2n+2} \lambda_k^{(1)} (x - \gamma)^{k-1} + O_n\left(\frac{T|x - \gamma|^{2n+2}}{M^{2n+3}}\right),$$

and for  $2 \leq i \leq 2n+2$

$$(3.3) \quad f^{(i)}(x) = \sum_{k=i}^{2n+2} \lambda_k^{(i)} (x - \gamma)^{k-i} + O_n\left(\frac{T|x - \gamma|^{2n+3-i}}{M^{2n+3}}\right),$$

where

$$(3.4) \quad \lambda_k = \frac{f^{(k)}(\gamma)}{k!} \text{ for } k = 2, \dots, 2n+2,$$

$$(3.5) \quad \lambda_k^{(1)} = \frac{f^{(k)}(\gamma)}{(k-1)!} = k\lambda_k \text{ for } k = 2, \dots, 2n+2,$$

and for  $2 \leq i \leq 2n+2$

$$(3.6) \quad \lambda_k^{(i)} = \frac{f^{(k)}(\gamma)}{(k-i)!} = \frac{k!}{(k-i)!} \lambda_k \text{ for } k = i, \dots, 2n+2.$$

Now we change variables from  $x$  to  $y = h(x - \gamma)$  by

$$(3.7) \quad f(x) - f(\gamma) = \lambda_2 h^2(x - \gamma),$$



such that  $y = h(x - \gamma)$  has the same sign as that of  $x - \gamma$ . We will choose a constant  $r$  later, and consider  $-r \leq y \leq r$  with  $r < M$ . In fact, on application we can take  $r = o(M)$ . By (3.1) we know that

$$\begin{aligned} y^2 &= \sum_{k=2}^{2n+2} \frac{\lambda_k}{\lambda_2} (x - \gamma)^k + O_n\left(\frac{|x - \gamma|^{2n+3}}{M^{2n+1}}\right) \\ &= (x - \gamma)^2 \left(1 + \sum_{k=3}^{2n+2} \frac{\lambda_k}{\lambda_2} (x - \gamma)^{k-2} + O_n\left(\frac{|x - \gamma|^{2n+1}}{M^{2n+1}}\right)\right), \end{aligned}$$

so we can get

$$(3.8) \quad y = (x - \gamma) \left(1 + \sum_{k=1}^{2n} \frac{\lambda_{k+2}}{\lambda_2} (x - \gamma)^k + O_n\left(\frac{|x - \gamma|^{2n+1}}{M^{2n+1}}\right)\right)^{\frac{1}{2}}.$$

For  $1 \leq j \leq 2n + 1$ , we have

$$\begin{aligned} (3.9) \quad y^j &= (x - \gamma)^j \left(1 + \sum_{k=1}^{2n} \frac{\lambda_{k+2}}{\lambda_2} (x - \gamma)^k + O_n\left(\frac{|x - \gamma|^{2n+1}}{M^{2n+1}}\right)\right)^{j/2} \\ &= (x - \gamma)^j \left(1 + \sum_{k=1}^{2n} \mu_{jk} (x - \gamma)^k + O_n\left(\frac{|x - \gamma|^{2n+1}}{M^{2n+1}}\right)\right) \end{aligned}$$

in Taylor expansion at  $x = \gamma$ . To determine the Taylor coefficients  $\mu_{jk}$ , we derive from (3.9) that

$$y^j = (x - \gamma)^j \left(1 + \sum_{i=1}^{\infty} C_{ji} \left(\sum_{k=1}^{2n} \frac{\lambda_{k+2}}{\lambda_2} (x - \gamma)^k + O_n\left(\frac{|x - \gamma|^{2n+1}}{M^{2n+1}}\right)\right)^i\right)$$

with coefficients  $C_{ji}$  of the binomial expansion and get

$$(3.10) \quad \mu_{j0} = 1, \quad \mu_{jk} = \sum_{i=1}^k \frac{C_{ji}}{\lambda_2^i} \sum_{\substack{3 \leq n_1, \dots, n_i \leq 2n+3 \\ n_1 + \dots + n_i = k+2i}} \lambda_{n_1} \cdots \lambda_{n_i} \text{ for } 1 \leq k \leq 2n.$$

The variable change between  $x$  and  $y$  in (3.9) allows us to compute  $f^{(i)}(x)$  for  $1 \leq i \leq 2n + 2$ . We have

**Lemma 3.1.** *With the above notation we assume (3.54) and (3.55). Then*

$$(3.11) \quad f'(x) = \sum_{k=1}^{2n+1} \theta_k^{(1)} y^k + O_n\left(\frac{T|y|^{2n+2}}{M^{2n+3}}\right),$$

where

$$(3.12) \quad \theta_1^{(1)} = \lambda_2^{(1)} = 2\lambda_2, \quad \theta_k^{(1)} = \sum_{j=1}^{k-1} \frac{C_{k,j}^{(1)}}{\lambda_2^{j-1}} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = k-1+2j}} \lambda_{n_1} \cdots \lambda_{n_j} \text{ for } 2 \leq k \leq 2n+1.$$

*Proof.* We claim that for any  $1 \leq m \leq 2n + 1$

$$(3.13) \quad f'(x) = \sum_{k=1}^m \theta_{m,k} y^k + \sum_{k=m+1}^{2n+1} \theta_{m,k} (x - \gamma)^k + O_n\left(\frac{T|x - \gamma|^{2n+2}}{M^{2n+3}}\right),$$

where

$$\theta_{m,1} = \lambda_2^{(1)} = 2\lambda_2, \quad \theta_{m,k} = \sum_{1 \leq j \leq k-1} \frac{C_{k,j}}{\lambda_2^{j-1}} \sum_{\substack{n_1 + \dots + n_j = k-1+2j \\ 3 \leq n_1, \dots, n_j \leq 2n+3}} \lambda_{n_1} \cdots \lambda_{n_j} \text{ for } 2 \leq k \leq 2n+1.$$

To prove (3.13) by induction we see that if substitute (3.9) in (3.2) for  $j = 1$ , then we get

$$\begin{aligned}
 (3.14) \quad f'(x) &= \lambda_2^{(1)} y - \lambda_2^{(1)} \sum_{k=1}^{2n} \mu_{1k} (x - \gamma)^{k+1} + \sum_{k=3}^{2n+2} \lambda_k^{(1)} (x - \gamma)^{k-1} + O_n\left(\frac{T|x - \gamma|^{2n+2}}{M^{2n+3}}\right) \\
 &= \lambda_2^{(1)} y + \sum_{k=2}^{2n+1} \left( \lambda_{k+1}^{(1)} - \lambda_2^{(1)} \mu_{1,k-1} \right) (x - \gamma)^k + O_n\left(\frac{T|x - \gamma|^{2n+2}}{M^{2n+3}}\right) \\
 &:= \theta_{1,1} y + \sum_{k=2}^{2n+1} \theta_{1,k} (x - \gamma)^k + O_n\left(\frac{T|x - \gamma|^{2n+2}}{M^{2n+3}}\right),
 \end{aligned}$$

which is (3.13) for  $m = 1$ . We see that for  $2 \leq k \leq 2n + 1$

$$\begin{aligned}
 \theta_{1,k} &= \lambda_{k+1}^{(1)} - \lambda_2^{(1)} \mu_{1,k-1} \\
 &= (k+1)\lambda_{k+1} - 2\lambda_2 \sum_{1 \leq j \leq k-1} \frac{C_{1,j}}{\lambda_2^j} \sum_{\substack{n_1 + \dots + n_j = k-1+2j \\ 3 \leq n_1, \dots, n_j \leq 2n+3}} \lambda_{n_1} \dots \lambda_{n_j} \\
 &= \sum_{1 \leq j \leq k-1} \frac{C'_{1,j}}{\lambda_2^{j-1}} \sum_{\substack{n_1 + \dots + n_j = k-1+2j \\ 3 \leq n_1, \dots, n_j \leq 2n+3}} \lambda_{n_1} \dots \lambda_{n_j}.
 \end{aligned}$$

Therefore for  $1 \leq k \leq 2n + 1$

$$\theta_{1,k} = \sum_{1 \leq j \leq k-1} \frac{C'''_{1,j}}{\lambda_2^{j-1}} \sum_{\substack{n_1 + \dots + n_j = k-1+2j \\ 3 \leq n_1, \dots, n_j \leq n+3}} \lambda_{n_1} \dots \lambda_{n_j},$$

which is the case for  $m = 1$ .

Assume that (3.13) holds for  $m = \ell$ . Then

$$(3.15) \quad f'(x) = \sum_{k=1}^{\ell} \theta_{\ell,k} y^k + \sum_{k=\ell+1}^{2n+1} \theta_{\ell,k} (x - \gamma)^k + O_n\left(\frac{T|x - \gamma|^{2n+2}}{M^{2n+3}}\right),$$

with

$$\theta_{\ell,k} = \sum_{1 \leq j \leq k-1} \frac{C_{\ell,j}}{\lambda_2^{j-1}} \sum_{\substack{n_1 + \dots + n_j = k-1+2j \\ 3 \leq n_1, \dots, n_j \leq 2n+3}} \lambda_{n_1} \dots \lambda_{n_j}.$$

Substitute (3.9) in (3.15) and note that  $\theta_{\ell,k} \ll_l \frac{T}{M^{k+\ell}}$ , then

$$\begin{aligned}
 (3.16) \quad f'(x) &= \sum_{k=1}^{\ell} \theta_{\ell,k} y^k + \theta_{\ell,\ell+1} y^{\ell+1} + \sum_{k=\ell+2}^{2n+1} \theta_{\ell,k} (x - \gamma)^k - \theta_{\ell,\ell+1} \sum_{k=1}^{2n} \mu_{\ell,k} (x - \gamma)^{k+\ell+1} \\
 &\quad + O_n\left(\frac{T|x - \gamma|^{2n+2}}{M^{2n+3}}\right) \\
 &= \sum_{k=1}^{\ell+1} \theta_{\ell,k} y^k + \sum_{k=\ell+2}^{2n+1} \left( \theta_{\ell,k} - \theta_{\ell,\ell+1} \mu_{\ell,k-\ell-1} \right) (x - \gamma)^k + O_n\left(\frac{T|x - \gamma|^{2n+2}}{M^{2n+3}}\right) \\
 &=: \sum_{k=1}^{\ell+1} \theta_{\ell+1,k} y^k + \sum_{k=\ell+2}^{2n+1} \theta_{\ell+1,k} (x - \gamma)^k + O_n\left(\frac{T|x - \gamma|^{2n+2}}{M^{2n+3}}\right)
 \end{aligned}$$

which is (3.13) for  $m = \ell + 1$ . For  $k \leq \ell + 1$

$$\theta_{\ell+1,k} = \theta_{\ell,k} = \sum_{1 \leq j \leq k-1} \frac{C_{\ell,j}}{\lambda_2^{j-1}} \sum_{\substack{n_1 + \dots + n_j = k-1+2j \\ 3 \leq n_1, \dots, n_j \leq 2n+3}} \lambda_{n_1} \dots \lambda_{n_j}$$

and for  $\ell + 2 \leq k \leq 2n + 1$  (ignoring all coefficients)

$$\begin{aligned}
\theta_{\ell+1,k} &= \theta_{\ell,k} - \theta_{\ell,\ell+1} \mu_{\ell,k-\ell-1} \\
&= \sum_{1 \leq j \leq k-1} \frac{1}{\lambda_2^{j-1}} \sum_{\substack{n_1 + \dots + n_j = k-1+2j \\ 3 \leq n_1, \dots, n_j \leq 2n+3}} \lambda_{n_1} \dots \lambda_{n_j} \\
&\quad + \sum_{1 \leq j \leq \ell} \frac{1}{\lambda_2^{j-1}} \sum_{\substack{n_1 + \dots + n_j = \ell+2j \\ 3 \leq n_1, \dots, n_j \leq 2n+3}} \lambda_{n_1} \dots \lambda_{n_j} \\
&\quad \times \sum_{1 \leq j \leq k-\ell-1} \frac{c}{\lambda_2^j} \sum_{\substack{n_1 + \dots + n_j = k-\ell-1+2j \\ 3 \leq n_1, \dots, n_j \leq 2n+3}} \lambda_{n_1} \dots \lambda_{n_j} \\
&= \sum_{1 \leq j \leq k-1} \frac{1}{\lambda_2^{j-1}} \sum_{\substack{n_1 + \dots + n_j = k-1+2j \\ 3 \leq n_1, \dots, n_j \leq 2n+3}} \lambda_{n_1} \dots \lambda_{n_j}.
\end{aligned}$$

Therefore for  $1 \leq k \leq 2n + 1$

$$\theta_{\ell+1,k} = \sum_{1 \leq j \leq k-1} \frac{C_{k,j}}{\lambda_2^{j-1}} \sum_{\substack{n_1 + \dots + n_j = k-1+2j \\ 3 \leq n_1, \dots, n_j \leq 2n+3}} \lambda_{n_1} \dots \lambda_{n_j}$$

which is the case for  $m = \ell + 1$ . We complete the proof of (3.13).

Now take  $m = 2n + 1$  in (3.13) we get

$$\begin{aligned}
f'(x) &= \sum_{k=1}^{2n+1} \theta_{2n+1,k} y^k + O_n\left(\frac{T|x-\gamma|^{2n+2}}{M^{2n+3}}\right) \\
&=: \sum_{k=1}^{2n+1} \theta_k^{(1)} y^k + O_n\left(\frac{T|x-\gamma|^{2n+2}}{M^{2n+3}}\right).
\end{aligned}$$

Using (3.7) and from the second order Talor expansion we see that

$$\lambda_2 y^2 = f(x) - f(\gamma) = \frac{f''(w)}{2!} (x - \gamma)^2,$$

where  $w$  is some constant between  $x$  and  $\gamma$ . Therefore

$$c_1 |x - \gamma| \leq y \leq c_2 |x - \gamma|.$$

Then using above estimates we get

$$f'(x) = \sum_{k=1}^{2n+1} \theta_k^{(1)} y^k + O_n\left(\frac{T|y|^{2n+2}}{M^{2n+3}}\right).$$

□

Similarly, we can change  $x$  to  $y$  in (3.3) by (3.9). We have for  $2 \leq i \leq 2n + 2$

$$(3.17) \quad f^{(i)}(x) = \sum_{k=0}^{2n+2-i} \theta_k^{(i)} y^k + O_n\left(\frac{T y^{2n+3-i}}{M^{2n+3}}\right),$$

where

$$(3.18) \quad \theta_0^{(i)} = \lambda_i^{(i)} = i! \lambda_i, \quad \theta_k^{(i)} = \sum_{j=1}^k \frac{C_{k,j}}{\lambda_2^{j-1}} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = k+i-2+2j}} \lambda_{n_1} \dots \lambda_{n_j} \text{ for } 1 \leq k \leq 2n + 2 - i.$$

Now by the definition of  $y$  in (3.7) we can compute  $\frac{dx}{dy}$ .

**Lemma 3.2.** *With the above notation we assume (3.54) and (3.55). Then*

$$(3.19) \quad \frac{dx}{dy} = \sum_{k=0}^{2n} \rho_k y^k + O_n\left(\frac{|y|^{2n+1}}{M^{2n+1}}\right)$$

where

$$(3.20) \quad \rho_0 = 1, \quad \rho_k = \sum_{j=1}^k \frac{C'_{kj}}{\lambda_2^j} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = k+2j}} \lambda_{n_1} \cdots \lambda_{n_j} \text{ for } k \geq 1.$$

*Proof.* First by (3.7) we see that

$$f'(x) = 2\lambda_2 y \frac{dy}{dx}$$

and

$$(3.21) \quad \frac{dx}{dy} = \frac{2\lambda_2 y}{f'(x)}.$$

Under the conditions (3.54) and (3.55), by Lemma 3.1 we see that

$$\begin{aligned} \frac{dx}{dy} &= \frac{2\lambda_2 y}{2\lambda_2 y + \sum_{k=2}^{2n+1} \theta_k^{(1)} y^k + O_n\left(\frac{T|y|^{2n+2}}{M^{2n+3}}\right)} \\ &= \frac{1}{1 + \sum_{k=2}^{2n+1} \frac{\theta_k^{(1)}}{2\lambda_2} y^{k-1} + O_n\left(\frac{|y|^{2n+1}}{M^{2n+1}}\right)} \\ &= \sum_{j=0}^{\infty} \left( - \sum_{k=2}^{2n+1} \frac{\theta_k^{(1)}}{2\lambda_2} y^{k-1} + O_n\left(\frac{|y|^{2n+1}}{M^{2n+1}}\right) \right)^j \\ &= \sum_{k=0}^{2n} \rho_k y^k + O_n\left(\frac{|y|^{2n+1}}{M^{2n+1}}\right) \end{aligned}$$

where (ignoring all coefficients)

$$(3.22) \quad \rho_0 = 1, \quad \rho_k = \sum_{j=1}^k C_{kj} \sum_{\substack{k_1, \dots, k_j \geq 1 \\ k_1 + \dots + k_j = k}} \frac{\theta_{k_1+1}^{(1)}}{\lambda_2} \cdots \frac{\theta_{k_j+1}^{(1)}}{\lambda_2} \text{ for } 1 \leq k \leq 2n.$$

Note that we used the following bounds

$$\theta_k^{(1)} \ll \frac{T}{M^{k+1}}, \quad \lambda_2 \ll \frac{T}{M^2}.$$

By (3.12) we see that for  $k_t, k_l \geq 1$  (ignoring all coefficients)

$$\begin{aligned} \frac{\theta_{k_t+1}^{(1)}}{\lambda_2} \cdot \frac{\theta_{k_l+1}^{(1)}}{\lambda_2} &= \sum_{1 \leq s \leq k_t} \frac{1}{\lambda_2^s} \sum_{\substack{3 \leq n_i \leq 2n+3 \\ n_1 + \dots + n_s = k_t + 2s}} \lambda_{n_1} \cdots \lambda_{n_s} \\ &\quad \times \sum_{1 \leq s \leq k_l} \frac{1}{\lambda_2^s} \sum_{\substack{3 \leq n_i \leq 2n+3 \\ n_1 + \dots + n_s = k_l + 2s}} \lambda_{n_1} \cdots \lambda_{n_s} \\ &= \sum_{1 \leq s \leq k_t + k_l} \frac{1}{\lambda_2^s} \sum_{\substack{n_1 + \dots + n_s = k_t + k_l + 2s \\ 3 \leq n_i \leq 2n+3}} \lambda_{n_1} \cdots \lambda_{n_s}. \end{aligned}$$

Therefore by (3.22) for  $k \geq 1$

$$\begin{aligned} \rho_k &= \sum_{j=1}^k C_{kj} \sum_{\substack{k_1, \dots, k_j \geq 1 \\ k_1 + \dots + k_j = k}} \sum_{1 \leq s \leq k_1 + \dots + k_j} \frac{1}{\lambda_2^s} \sum_{\substack{n_1 + \dots + n_s = k_1 + \dots + k_j + 2s \\ 3 \leq n_i \leq 2n+3}} \lambda_{n_1} \cdots \lambda_{n_s} \\ &= \sum_{s=1}^k \frac{C'_{ks}}{\lambda_2^s} \sum_{\substack{3 \leq n_1, \dots, n_s \leq 2n+3 \\ n_1 + \dots + n_s = k + 2s}} \lambda_{n_1} \cdots \lambda_{n_s}. \end{aligned}$$

□

Let  $g(x)$  be a real function,  $2n+1$  times continuously differentiable for  $\alpha \leq x \leq \beta$ . The Taylor expansions give

$$(3.23) \quad g(x) = \sum_{k=0}^{2n} \eta_k (x - \gamma)^k + O_n \left( \frac{U|x - \gamma|^{2n+1}}{N^{2n+1}} \right),$$

$$(3.24) \quad \frac{d^i g}{dx^i} = \sum_{k=i}^{2n} \eta_k^{(i)} (x - \gamma)^{k-i} + O_n \left( \frac{U|x - \gamma|^{2n+1-i}}{N^{2n+1}} \right),$$

where

$$(3.25) \quad \eta_0 = g(\gamma), \quad \eta_k = \frac{g^{(k)}(\gamma)}{k!} \text{ for } 1 \leq k \leq 2n,$$

$$(3.26) \quad \eta_k^{(i)} = \frac{g^{(k)}(\gamma)}{(k-i)!} = \frac{k!}{(k-i)!} \eta_k \text{ for } i \leq k \leq 2n.$$

Similarly, if we change variables in (3.23) to  $y$ , we can get

$$(3.27) \quad g(x) = \sum_{k=0}^{2n} \eta'_k y^k + O_n \left( U|y|^{2n+1} \left( \frac{1}{NM^{2n}} + \frac{1}{N^{2n+1}} \right) \right),$$

with

$$(3.28) \quad \eta'_0 = g(\gamma).$$

To determine other  $\eta'_k$ , we substitute (3.9) into (3.27) to get

$$\begin{aligned} g(x) &= \eta'_0 + \sum_{k=1}^{2n} \eta'_k (x - \gamma)^k \left( \sum_{\ell=0}^{2n} \mu_{k\ell} (x - \gamma)^\ell + O_n \left( \frac{|x - \gamma|^{2n+1}}{M^{2n+1}} \right) \right) \\ &\quad + O_n \left( U|y|^{2n+1} \left( \frac{1}{NM^{2n}} + \frac{1}{N^{2n+1}} \right) \right) \\ &= \eta'_0 + \sum_{m=1}^{2n} (x - \gamma)^m \sum_{\substack{k \geq 1 \\ \ell \geq 0 \\ k + \ell = m}} \eta'_k \mu_{k\ell} + O_n \left( U|y|^{2n+1} \left( \frac{1}{NM^{2n}} + \frac{1}{N^{2n+1}} \right) \right). \end{aligned}$$

Consequently

$$(3.29) \quad \eta_1 = \sum_{\substack{k \geq 1 \\ \ell \geq 0 \\ k + \ell = m}} \eta'_k \mu_{k\ell} = \eta'_1 \mu_{10} = \eta'_1,$$

and for  $2 \leq m \leq 2n$

$$(3.30) \quad \eta_m = \sum_{\substack{k \geq 1 \\ \ell \geq 0 \\ k + \ell = m}} \eta'_k \mu_{k\ell} = \eta'_m + \sum_{\substack{k, \ell \geq 1 \\ k + \ell = m}} \eta'_k \mu_{k\ell},$$

where  $\eta_m$  is given in (3.23). We may compute  $\eta'_m$  for  $m \geq 2$  recursively using (3.30).

**Lemma 3.3.** *With the above notation*

$$(3.31) \quad \eta'_m = \eta_m + \sum_{k=1}^{m-1} \eta_k \sum_{j=1}^{m-k} \frac{C_{mkj}}{\lambda_2^j} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = m-k+2j}} \lambda_{n_1} \cdots \lambda_{n_j}$$

where the  $k$  sum vanishes when  $m \leq 1$ .

*Proof.* By (3.29), (3.31) holds for  $m = 1$ . Suppose (3.31) holds for any number  $\leq m$ . Then by (3.30) and (3.10)

$$(3.32) \quad \begin{aligned} \eta'_{m+1} &= \eta_{m+1} - \sum_{\ell=1}^m \eta'_\ell \mu_{\ell, m+1-\ell} \\ &= \eta_{m+1} - \sum_{\ell=1}^m \eta_\ell \sum_{i=1}^{m+1-\ell} \frac{C_{\ell i}}{\lambda_2^i} \sum_{\substack{3 \leq m_1, \dots, m_i \leq 2n+3 \\ m_1 + \dots + m_i = m+1-\ell+2i}} \lambda_{m_1} \cdots \lambda_{m_i} \\ &\quad - \sum_{\ell=1}^m \sum_{k=1}^{\ell-1} \eta_k \sum_{j=1}^{\ell-k} \frac{C_{\ell k j}}{\lambda_2^j} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = \ell-k+2j}} \lambda_{n_1} \cdots \lambda_{n_j} \\ &\quad \times \sum_{i=1}^{m+1-\ell} \frac{C_{\ell i}}{\lambda_2^i} \sum_{\substack{3 \leq m_1, \dots, m_i \leq 2n+3 \\ m_1 + \dots + m_i = m+1-\ell+2i}} \lambda_{m_1} \cdots \lambda_{m_i}. \end{aligned}$$

The first two terms on the right side of (3.32) fit (3.31) for  $m+1$ . For the third term, we change the order of sums on  $\ell$  and  $k$ , let  $h = i + j$ , denote  $m_1, \dots, m_i, n_1, \dots, n_j$  by  $p_1, \dots, p_h$ , and get

$$- \sum_{k=1}^{m-1} \eta_k \sum_{\ell=k+1}^m \sum_{h=2}^{m+1-k} \frac{1}{\lambda_2^h} \sum_{\substack{i, j \geq 1 \\ i+j=h}} C_{\ell i} C_{\ell k j} \sum_{\substack{3 \leq p_1, \dots, p_h \leq 2n+3 \\ p_1 + \dots + p_h = m+1-k+2h \\ p_1 + \dots + p_i = m+1-\ell+2i}} \lambda_{p_1} \cdots \lambda_{p_h}$$

which also fits (3.31) for  $m+1$ . □

Similarly using (3.9) in (3.24) we get for  $1 \leq i \leq n+1$

$$(3.33) \quad \frac{d^i g}{dx^i} = \sum_{k=0}^{2n-i} \eta_k^{(i)'} y^k + O_n \left( U|y|^{2n+1-i} \left( \frac{1}{NM^{2n}} + \frac{1}{N^{2n+1}} \right) \right),$$

where

$$(3.34) \quad \eta_k^{(i)'} = \frac{(k+i)!}{k!} \eta_{k+i} + \sum_{m=1}^{k+i-1} \eta_m \sum_{j=1}^{k+i-m} \frac{C_{kmj}}{\lambda_2^j} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = k+i-m+2j}} \lambda_{n_1} \cdots \lambda_{n_j}.$$

Multiplying  $g(x)$  in (3.27) with  $\frac{dx}{dy}$  in (3.19) and using

$$\eta_k^{(i)'} \ll \left( \frac{U}{N^k} + \frac{U}{NM^{k-1}} \right), \quad \rho_k \ll \frac{1}{M^k},$$

we get

$$(3.35) \quad g(x) \frac{dx}{dy} = \sum_{k=0}^{2n} \varpi_k y^k + O_n \left( U|y|^{2n+1} \left( \frac{1}{M^{2n+1}} + \frac{1}{N^{2n+1}} \right) \right),$$

where

$$(3.36) \quad \varpi_k = \sum_{\ell=0}^k \eta'_\ell \rho_{k-\ell}.$$

Note that

$$(3.37) \quad \varpi_0 = \eta'_0 \rho_0 = g(\gamma)$$

by (3.20) and (3.28).

**Lemma 3.4.** *With the notation as above*

$$(3.38) \quad \varpi_k = \eta_k + \sum_{\ell=0}^{k-1} \eta_\ell \sum_{j=1}^{k-\ell} \frac{C_{k\ell j}}{\lambda_2^j} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = k - \ell + 2j}} \lambda_{n_1} \cdots \lambda_{n_j},$$

for some constant coefficients  $C_{klj}$ .

*Proof.* By (3.36), (3.31) and (3.20) we have

$$(3.39) \quad \varpi_N = \sum_{m=0}^{N-1} \left( \eta_m + \sum_{k=1}^{m-1} \eta_k \sum_{j=1}^{m-k} \frac{C_{mkj}}{\lambda_2^j} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = m - k + 2j}} \lambda_{n_1} \cdots \lambda_{n_j} \right)$$

$$(3.40) \quad \times \sum_{h=1}^{N-m} \frac{C_{N-m,h}}{\lambda_2^h} \sum_{\substack{3 \leq m_1, \dots, m_h \leq 2n+3 \\ m_1 + \dots + m_h = N - m + 2h}} \lambda_{m_1} \cdots \lambda_{m_h}$$

$$(3.41) \quad + \eta_N + \sum_{k=1}^{N-1} \eta_k \sum_{j=1}^{N-k} \frac{C_{Nkj}}{\lambda_2^j} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = N - k + 2j}} \lambda_{n_1} \cdots \lambda_{n_j},$$

where the first  $k$  sum vanishes when  $m \leq 1$ . Note that (3.41) is contained in (3.38), while  $\sum_{m=0}^{N-1} \eta_m$  times (3.40) is also contained in (3.38). The rest sum of  $\varpi_N$  equals

$$\begin{aligned} & \sum_{m=2}^{N-1} \sum_{k=1}^{m-1} \eta_k \sum_{j=1}^{m-k} \frac{C_{mkj}}{\lambda_2^j} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = m - k + 2j}} \lambda_{n_1} \cdots \lambda_{n_j} \\ & \times \sum_{h=1}^{N-m} \frac{C_{N-m,h}}{\lambda_2^h} \sum_{\substack{3 \leq m_1, \dots, m_h \leq 2n+3 \\ m_1 + \dots + m_h = N - m + 2h}} \lambda_{m_1} \cdots \lambda_{m_h} \\ & = \sum_{k=1}^{N-2} \eta_k \sum_{m=k+1}^{N-1} \sum_{i=1}^{N-k} \frac{1}{\lambda_2^i} \sum_{\substack{h,j \geq 1 \\ h+j=i}} C_{mkj} C_{N-m,h} \sum_{\substack{3 \leq p_1, \dots, p_i \leq 2n+3 \\ p_1 + \dots + p_i = N - k + 2i \\ p_1 + \dots + p_h = N - m + 2h}} \lambda_{p_1} \cdots \lambda_{p_i}, \end{aligned}$$

which also fits (3.38). □

For example for  $n = 2$ , if we ignore all coefficients we have

$$(3.42) \quad \varpi_1 = g'(\gamma) + g(\gamma) \frac{f^{(3)}(\gamma)}{f''(\gamma)},$$

$$(3.43) \quad \varpi_2 = g''(\gamma) + g'(\gamma) \frac{f^{(3)}(\gamma)}{f''(\gamma)} + g(\gamma) \left( \frac{f^{(4)}(\gamma)}{f''(\gamma)} + \frac{f^{(3)}(\gamma)^2}{f''(\gamma)^2} \right),$$

$$(3.44) \quad \begin{aligned} \varpi_3 = & g^{(3)}(\gamma) + g''(\gamma) \frac{f^{(3)}(\gamma)}{f''(\gamma)} + g'(\gamma) \left( \frac{f^{(4)}(\gamma)}{f''(\gamma)} + \frac{f^{(3)}(\gamma)^2}{f''(\gamma)^2} \right) \\ & + g(\gamma) \left( \frac{f^{(5)}(\gamma)}{f''(\gamma)} + \frac{f^{(4)}(\gamma)f^{(3)}(\gamma)}{f''(\gamma)^2} + \frac{f^{(3)}(\gamma)^3}{f''(\gamma)^3} \right), \end{aligned}$$

and

$$(3.45) \quad \begin{aligned} \varpi_4 = & g^{(4)}(\gamma) + g^{(3)}(\gamma) \frac{f^{(3)}(\gamma)}{f''(\gamma)} + g''(\gamma) \left( \frac{f^{(4)}(\gamma)}{f''(\gamma)} + \frac{f^{(3)}(\gamma)^2}{f''(\gamma)^2} \right) \\ & + g'(\gamma) \left( \frac{f^{(5)}(\gamma)}{f''(\gamma)} + \frac{f^{(4)}(\gamma)f^{(3)}(\gamma)}{f''(\gamma)^2} + \frac{f^{(3)}(\gamma)^3}{f''(\gamma)^3} \right) \\ & + g(\gamma) \left( \frac{f^{(6)}(\gamma)}{f''(\gamma)} + \frac{f^{(5)}(\gamma)f^{(3)}(\gamma)}{f''(\gamma)^2} + \frac{f^{(4)}(\gamma)^2}{f''(\gamma)^2} + \frac{f^{(4)}(\gamma)f^{(3)}(\gamma)^2}{f''(\gamma)^3} + \frac{f^{(3)}(\gamma)^4}{f''(\gamma)^4} \right). \end{aligned}$$

In the following lemma we compute derivatives of  $g(x) \frac{dx}{dy}$  which we will use to prove our main theorem.

**Lemma 3.5.** *With the above notation we assume (3.54) and (3.55). Then for  $1 \leq i \leq n+2$*

$$(3.46) \quad \frac{d^i}{dy^i} \left( g(x) \frac{dx}{dy} \right) = \sum_{k=0}^{2n-i} \frac{(k+i)!}{k!} \varpi_{k+i} y^k + O_n \left( U|y|^{2n+1-i} \left( \frac{1}{M^{2n+1}} + \frac{1}{N^{2n+1}} \right) \right).$$

*Proof.* By (3.38) we see that the expression of  $\varpi_k$ ,  $0 \leq k \leq 2n$ , only uses  $\eta_\ell$ ,  $0 \leq \ell \leq 2n$ , and hence it only uses  $g^{(\ell)}(\gamma)$  for  $0 \leq \ell \leq 2n$  by (3.25). By the same (3.38),  $\varpi_k$ ,  $0 \leq k \leq 2n$ , only requires  $\lambda_{n_1}, \dots, \lambda_{n_j}$  for  $n_1, \dots, n_j \leq 2n+2$ . Thus by (3.4), it only requires  $f^{(\ell)}(\gamma)$ ,  $\ell = 2, \dots, 2n+2$ . Consequently,  $\varpi_k$  for  $0 \leq k \leq 2n$  are independent of  $y$ , and the terms

$$\sum_{k=0}^{2n} \varpi_k y^k$$

are the corresponding terms in the Taylor expansion of  $g(x) \frac{dx}{dy}$ .

This implies that

$$(3.47) \quad \frac{d^i}{dy^i} \left( g(x) \frac{dx}{dy} \right) = \sum_{k=0}^{2n-i} \frac{(k+i)!}{k!} \varpi_{k+i} y^k + R_i(y).$$

where  $R_i(y)$  is the remainder term. We want to show

$$(3.48) \quad R_i(y) \ll O_n \left( U|y|^{2n+1-i} \left( \frac{1}{M^{2n+1}} + \frac{1}{N^{2n+1}} \right) \right).$$

In the following we will only consider the case of  $i = 1$ . Other cases are similar.

From (3.33) and (3.34)

$$(3.49) \quad \frac{dg}{dx} = \sum_{k=0}^{2n-1} \eta_k^{(1)'} y^k + O_n \left( U|y|^{2n} \left( \frac{1}{NM^{2n}} + \frac{1}{N^{2n+1}} \right) \right),$$

where

$$(3.50) \quad \eta_k^{(1)'} = \frac{(k+1)!}{k!} \eta_{k+1} + \sum_{m=1}^k \eta_m \sum_{j=1}^{k+1-m} \frac{C_{kmj}}{\lambda_2^j} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = k+1-m+2j}} \lambda_{n_1} \dots \lambda_{n_j}$$

By (3.21) we see that

$$\frac{d^2 x}{dy^2} = 2\lambda_2 \frac{1}{f'} - 2\lambda_2 y \frac{f''}{(f')^2} \frac{dx}{dy} = \frac{1}{y} \left( \frac{dx}{dy} - \frac{1}{2\lambda_2} \left( \frac{dx}{dy} \right)^3 f'' \right).$$



From last equation, (3.17) for  $i = 2$ , and (3.19) we see that  $\frac{d^2x}{dy^2}$  can be expressed as power series of  $y$ , i.e.

$$(3.51) \quad \frac{d^2x}{dy^2} = \sum_{k=0}^{2n-1} \rho_k^{(1)} y^k + O\left(\frac{|y|^{2n}}{M^{2n+1}}\right)$$

where

$$(3.52) \quad \rho_k^{(1)} = \sum_{j=1}^{k+1} \frac{C_{kj}}{\lambda_2^j} \sum_{\substack{3 \leq n_1, \dots, n_j \leq 2n+3 \\ n_1 + \dots + n_j = k+1+2j}} \lambda_{n_1} \cdots \lambda_{n_j}.$$

With these preparations, we compute

$$\frac{d}{dy} \left( g(x) \frac{dx}{dy} \right) = \frac{dg}{dx} \left( \frac{dx}{dy} \right)^2 + g(x) \frac{d^2x}{dy^2}.$$

By (3.50), (3.19), (3.27) and (3.52) we get

$$\frac{d}{dy} \left( g(x) \frac{dx}{dy} \right) = \sum_{m=0}^{2n-1} \varpi_m^{(1)} y^m + O_n \left( U |y|^{2n} \left( \frac{1}{M^{2n+1}} + \frac{1}{N^{2n+1}} \right) \right)$$

with

$$\varpi_m^{(1)} = \sum_{\substack{k, \ell_1, \ell_2 \geq 0 \\ k + \ell_1 + \ell_2 = m}} \eta_k^{(1)'} \rho_{\ell_1} \rho_{\ell_2} + \sum_{\substack{k, \ell \geq 0 \\ k + \ell = m}} \eta_k^{(1)'} \rho_\ell^{(1)}.$$

For  $0 \leq m \leq 2n-1$ ,  $\varpi_m^{(1)}$  as above involves  $\eta_\ell$ ,  $0 \leq \ell \leq 2n$ , and  $\lambda_{n_1}, \dots, \lambda_{n_j}$  with  $3 \leq n_1, \dots, n_j \leq 2n+2$ , and hence is independent of  $y$ . Consequently, the terms for  $0 \leq m \leq 2n-1$  in (3.53) are terms in the Taylor expansion of  $\frac{d}{dy} \left( g(x) \frac{dx}{dy} \right)$ . Comparing this with the Taylor terms in (3.47) for  $i = 1$ , we conclude that for  $0 \leq k \leq 2n-1$

$$(k+1)\varpi_{k+1} = \varpi_k^{(1)},$$

by the uniqueness of Taylor expansions. Therefore we see that (3.48) holds for  $i = 1$ .  $\square$

Define  $H_k(x)$  as in (2.4) for  $k = 1, \dots, n$ . Then they can be expressed as in (2.6).

**Theorem 3.6.** *Let  $f(x)$ ,  $g(x)$  and  $H_k(x)$  be defined as above with the same notations in (3.1)–(3.45). Suppose that there are positive parameters  $M, N, T, U$ , with*

$$(3.53) \quad M > \beta - \alpha,$$

and positive constants  $C_r$  such that for  $\alpha \leq x \leq \beta$ ,

$$(3.54) \quad |f^{(r)}(x)| \leq C_r \frac{T}{M^r}, \text{ for } r = 2, 3, \dots, 2n+3$$

$$(3.55) \quad f''(x) \geq \frac{T}{C_2 M^2}$$

and

$$(3.56) \quad |g^{(s)}(x)| \leq C_s \frac{U}{N^s}, \text{ for } s = 0, 1, 2, \dots, 2n+1.$$

If  $T$  is sufficiently large comparing to the constants  $C_r$ , we have for  $n \geq 2$  that

$$\begin{aligned}
(3.57) \quad & \int_{\alpha}^{\beta} g(x)e(f(x))dx \\
&= \frac{e\left(f(\gamma) + \frac{1}{8}\right)}{\sqrt{f''(\gamma)}} \left( g(\gamma) + \sum_{j=1}^n \varpi_{2j} \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \right) + \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_{\alpha}^{\beta} \\
&+ O\left( \frac{UM^{2n+5}}{NT^{n+2}} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \left( \frac{1}{(\gamma-\alpha)^{n+2+j}} + \frac{1}{(\beta-\gamma)^{n+2+j}} \right) \sum_{t=j}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right) \\
&+ O\left( \frac{UM^{2n+4}}{T^{n+2} N^{n+1}} \left( \frac{M}{N} + 1 \right) \left( \frac{1}{(\gamma-\alpha)^{n+2}} + \frac{1}{(\beta-\gamma)^{n+2}} \right) \right) \\
&+ O\left( \frac{UM^{2n+4}}{T^{n+2}} \sum_{j=1}^{n+1} \left( \frac{1}{(\gamma-\alpha)^{n+2+j}} + \frac{1}{(\beta-\gamma)^{n+2+j}} \right) \sum_{t=0}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right) \\
&+ O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).
\end{aligned}$$

Note that the last error term in (3.57) is

$$\ll UN \left( \frac{M^2}{TN^2} \right)^{n+1} + \frac{UM}{T^{n+1}}.$$

To get a smaller error bound for larger  $n$ , we need to assume

$$(3.58) \quad N \geq \frac{M^{1+\varepsilon}}{\sqrt{T}}.$$

In this case Theorem 3.6 sharpens Lemma 5.5.6 of [10], p.114, with main terms up to the  $n$ th order, more boundary terms and smaller error terms.

We may define

$$\begin{aligned}
(3.59) \quad f(x) &= f(\alpha) + \sum_{k=1}^{2n+3} \frac{f^{(k)}(\alpha)}{k!} (x-\alpha)^k \text{ for } x < \alpha; \\
&= f(\beta) + \sum_{k=1}^{2n+3} \frac{f^{(k)}(\beta)}{k!} (x-\alpha)^k \text{ for } x > \beta,
\end{aligned}$$

and

$$\begin{aligned}
(3.60) \quad g(x) &= g(\alpha) + \sum_{k=1}^{2n+1} \frac{g^{(k)}(\alpha)}{k!} (x-\alpha)^k \text{ for } x < \alpha; \\
&= g(\beta) + \sum_{k=1}^{2n+1} \frac{g^{(k)}(\beta)}{k!} (x-\alpha)^k \text{ for } x > \beta.
\end{aligned}$$

Then on an interval  $[\alpha - cM, \beta + cM]$  for a positive constant  $c$ , the inequalities (3.54)–(3.56) still hold with possibly larger constants  $C_r$ . Using this enlarged interval

$$(3.61) \quad \frac{1}{\gamma - \alpha} \asymp \frac{1}{M}, \quad \frac{1}{\beta - \gamma} \asymp \frac{1}{M}$$

as they appear in (3.57).

#### 4. THE PROOF OF THEOREM 3.6

Recall (3.7). We choose a positive  $r \leq cM$  for some positive constant  $c < 1$  and define  $u, v$  by

$$h(u - \gamma) = -r, \quad h(v - \gamma) = r,$$

i.e.,

$$f(u) = f(v) = f(\gamma) + \lambda_2 r^2.$$

By Theorem 2.1 with  $n$  changing to  $n+1$ , we have for  $\alpha \leq u$

$$\begin{aligned} & \int_{\alpha}^u g(x) e(f(x)) dx \\ = & \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_{\alpha}^u + O\left( \frac{M}{N} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{UT^j}{\min |f'|^{n+j+2} M^{2j}} \sum_{t=j}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right) \\ & + O\left( \left( \frac{M}{N} + 1 \right) \frac{U}{N^{n+1} \min |f'|^{n+2}} + \sum_{j=1}^{n+1} \frac{UT^j}{\min |f'|^{n+j+2} M^{2j}} \sum_{t=0}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right). \end{aligned}$$

Since

$$f(u) - f(\gamma) = \frac{f''(\eta)}{2!} |u - \gamma|^2 = \lambda_2 r^2$$

for some constant  $\eta \in (\alpha, u)$ , we see that

$$c_1 r \leq |u - \gamma| \leq c_2 r$$

for some positive constants  $c_1$  and  $c_2$ . Then for  $\alpha \leq x \leq u$

$$|f'(x)| \geq \frac{T|x - \gamma|}{C_2 M^2} \geq \frac{T|u - \gamma|}{C_2 M^2} \geq \frac{Tr}{C_2' M^2}.$$

Therefore for  $\alpha \leq u$

$$\begin{aligned} (4.1) \quad & \int_{\alpha}^u g(x) e(f(x)) dx \\ = & \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_{\alpha}^u + O\left( \frac{UM^{2n+5}}{NT^{n+2}r^{n+2}} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{r^j} \sum_{t=j}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right) \\ & + O\left( \frac{UM^{2n+4}}{T^{n+2}r^{n+2}} \left[ \frac{1}{N^{n+1}} \left( \frac{M}{N} + 1 \right) + \sum_{j=1}^{n+1} \frac{1}{r^j} \sum_{t=0}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right] \right). \end{aligned}$$

Similarly for  $u \leq \alpha$  we have

$$\begin{aligned} (4.2) \quad & \int_u^{\alpha} g(x) e(f(x)) dx \\ = & \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_u^{\alpha} \\ & + O\left( \frac{UM^{2n+5}}{NT^{n+2}(\gamma - \alpha)^{n+2}} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{(\gamma - \alpha)^j} \sum_{t=j}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right) \\ & + O\left( \frac{UM^{2n+4}}{T^{n+2}(\gamma - \alpha)^{n+2}} \left[ \frac{1}{N^{n+1}} \left( \frac{M}{N} + 1 \right) + \sum_{j=1}^{n+1} \frac{1}{(\gamma - \alpha)^j} \sum_{t=0}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right] \right), \end{aligned}$$

because for  $u \leq x \leq \alpha$

$$|f'(x)| \geq \frac{T|\alpha - \gamma|}{C_2 M^2}.$$

In the same way, we have for  $v \leq \beta$

$$\begin{aligned}
 (4.3) \quad & \int_v^\beta g(x)e(f(x))dx \\
 &= \left[ e(f(x)) \cdot \sum_{i=1}^n H_i(x) \right]_v^\beta + O\left( \frac{UM^{2n+5}}{NT^{n+2}r^{n+2}} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{r^j} \sum_{t=j}^{n+1-j} \frac{1}{N^{n+1-j-t}M^t} \right) \\
 &+ O\left( \frac{UM^{2n+4}}{T^{n+2}r^{n+2}} \left[ \frac{1}{N^{n+1}} \left( \frac{M}{N} + 1 \right) + \sum_{j=1}^{n+1} \frac{1}{r^j} \sum_{t=0}^{n+1-j} \frac{1}{N^{n+1-j-t}M^t} \right] \right),
 \end{aligned}$$

and for  $\beta \leq v$

$$\begin{aligned}
 (4.4) \quad & \int_\beta^v g(x)e(f(x))dx \\
 &= \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_\beta^v \\
 &+ O\left( \frac{UM^{2n+5}}{NT^{n+2}(\beta-\gamma)^{n+2}} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{(\beta-\gamma)^j} \sum_{t=j}^{n+1-j} \frac{1}{N^{n+1-j-t}M^t} \right) \\
 &+ O\left( \frac{UM^{2n+4}}{T^{n+2}(\beta-\gamma)^{n+2}} \left[ \frac{1}{N^{n+1}} \left( \frac{M}{N} + 1 \right) + \sum_{j=1}^{n+1} \frac{1}{(\beta-\gamma)^j} \sum_{t=0}^{n+1-j} \frac{1}{N^{n+1-j-t}M^t} \right] \right).
 \end{aligned}$$

By (4.1)–(4.4) we need to consider

$$\begin{aligned}
 (4.5) \quad & \int_u^v g(x)e(f(x))dx = e(f(\gamma)) \int_{-r}^r e(\lambda_2 y^2) g(x) \frac{dx}{dy} dy \\
 &= e(f(\gamma)) \int_{-r}^r e(\lambda_2 y^2) \sum_{k=0}^{\infty} \varpi_k y^k dy \\
 &= e(f(\gamma)) \sum_{k=0}^{2n} \varpi_k \int_{-r}^r e(\lambda_2 y^2) y^k dy + e(f(\gamma)) \int_{-r}^r e(\lambda_2 y^2) Q(y) dy,
 \end{aligned}$$

where

$$Q(y) = g(x) \frac{dx}{dy} - \sum_{k=0}^{2n} \varpi_k y^k,$$

because  $\varpi_k$ ,  $0 \leq k \leq 2n$ , is independent of  $y$ . To estimate (4.5), we need to bound  $\varpi_k$ . By (3.38), (3.54)–(3.56)

$$\begin{aligned}
 \varpi_k &\ll \sum_{l=0}^k \frac{U}{N^l} \sum_{j=1}^{k-l} \frac{M^{2j}}{T^j} \frac{T^j}{M^{k-l+2j}} \\
 &\ll \frac{U}{M^k} \sum_{l=0}^k \left( \frac{M}{N} \right)^l \\
 &\ll \frac{U}{M^k} \left( 1 + \left( \frac{M}{N} \right)^k \right).
 \end{aligned}$$

Consequently

$$(4.6) \quad \varpi_k \ll U \left( \frac{1}{N^k} + \frac{1}{M^k} \right) \text{ for } 1 \leq k \leq 2n.$$

Now we estimate the first integral on the right hand side of (4.5). It is easy to see that

$$\int_{-r}^r e(\lambda_2 y^2) y^k dy = 0$$

if  $k$  is odd. Now let  $k = 2j$  be even. Using integration by parts for  $j$  times we get

$$(4.7) \quad \int_{-r}^r e(\lambda_2 y^2) y^k dy = \left[ e(\lambda_2 y^2) \cdot \sum_{i=1}^j \phi_i^{(k)}(y) \right]_{-r}^r - \int_{-r}^r e(f(y)) (\phi_j^{(k)}(y))' dy,$$

where

$$(4.8) \quad \phi_1^{(k)}(y) = \frac{y^{k-1}}{4\pi\lambda_2 i}, \quad \phi_t^{(k)}(y) = -\frac{(\phi_{t-1}^{(k)}(y))'}{4\pi\lambda_2 i y} \text{ for } 2 \leq t \leq j.$$

We also define  $\phi_t^{(k)}(y)$  for  $j+1 \leq t \leq n+1$  just using the same structure, i.e.,

$$(4.9) \quad \phi_t^{(k)}(y) = -\frac{(\phi_{t-1}^{(k)}(y))'}{4\pi\lambda_2 i y} \text{ for } j+1 \leq t \leq n+1.$$

Note that we can not get those terms for  $j+1 \leq t \leq n+1$  directly by partial integration. By induction we get

$$(4.10) \quad \phi_t^{(k)}(y) = (-1)^{t-1} \frac{(2j-1) \cdots (2j-(2t-3)) \cdot y^{2j-(2t-1)}}{(4\pi\lambda_2 i)^t}, \quad 2 \leq t \leq j,$$

$$(4.11) \quad \phi_t^{(k)}(y) = \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^t y}, \quad t = j+1,$$

and

$$(4.12) \quad \phi_t^{(k)}(y) = \frac{(-1)^j (2j-1)!! (2t-2j-3)!!}{(4\pi i \lambda_2)^t r^{2(t-j)-1}}, \quad j+2 \leq t \leq n+1.$$

In particular, we have

$$(4.13) \quad \phi_j^{(k)}(y) = (-1)^{j-1} \frac{(2j-1)!! \cdot y}{(4\pi\lambda_2 i)^j}, \quad (\phi_j^{(k)}(y))' = (-1)^{j-1} \frac{(2j-1)!!}{(4\pi\lambda_2 i)^j}.$$

Actually, for odd  $k < 2n$ , we can also define  $\phi_t^{(k)}(y)$  for  $1 \leq t \leq n+1$  as above. And by symmetry we have for odd  $k$

$$(4.14) \quad \left[ e(\lambda_2 y^2) \cdot \sum_{i=1}^{n+1} \phi_i^{(k)}(y) \right]_{-r}^r = 0.$$

Now we continue to compute the case for even  $k = 2j$ . By (4.7) and (4.13) we get

$$(4.15) \quad \int_{-r}^r e(\lambda_2 y^2) y^k dy = \left[ e(\lambda_2 y^2) \cdot \sum_{i=1}^j \phi_i^{(k)}(y) \right]_{-r}^r + \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \cdot 2 \int_0^r e(\lambda_2 y^2) dy.$$

The integral on the right side of (4.15) can be expressed in terms of the probability integral (cf. Gradshteyn and Ryzhik [7] 8.251.1)

$$\Phi(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} \frac{e^{-t}}{\sqrt{t}} dt.$$

In fact,

$$(4.16) \quad 2 \int_0^r e(\lambda_2 y^2) dy = \frac{2}{\sqrt{2\pi\lambda_2}} \int_0^x e^{it^2} dt = \frac{e(\frac{1}{8})}{\sqrt{2\lambda_2}} \Phi\left(\frac{x}{e(1/8)}\right)$$

for  $x = r\sqrt{2\pi\lambda_2}$ , by [7] 8.256.1. An asymptotic expansion of (4.16) is given by [7] 8.254

$$\begin{aligned} 2 \int_0^r e(\lambda_2 y^2) dy &= \frac{e(\frac{1}{8})}{\sqrt{2\lambda_2}} \left( 1 - \frac{e^{ix^2} e(\frac{1}{8})}{x\sqrt{\pi}} \left[ \sum_{k=0}^d (-1)^k \frac{(2k-1)!!}{(2x^2/i)^k} + O(x^{-2d-2}) \right] \right) \\ &= \frac{e(\frac{1}{8})}{\sqrt{2\lambda_2}} + \frac{e^{ix^2}}{\sqrt{2\pi\lambda_2} ix} \sum_{k=0}^d \frac{(2k-1)!!}{(2ix^2)^k} + O\left(\frac{1}{\sqrt{\lambda_2}} x^{-2d-3}\right). \end{aligned}$$

Back to  $r$ , we have

$$2 \int_0^r e(\lambda_2 y^2) dy = \frac{e(\frac{1}{8})}{\sqrt{f''(\gamma)}} + \frac{e(\lambda_2 r^2)}{2\pi i \lambda_2 r} \sum_{k=0}^d \frac{(2k-1)!!}{(4\pi i \lambda_2 r^2)^k} + O\left(\frac{1}{\lambda_2^{d+2} r^{2d+3}}\right).$$

Or

$$(4.17) \quad \int_{-r}^r e(\lambda_2 y^2) dy = \frac{e(\frac{1}{8})}{\sqrt{f''(\gamma)}} + \frac{e(\lambda_2 r^2)}{2\pi i \lambda_2 r} \left(1 + \sum_{k=2}^d \frac{(2k-3)!!}{(4\pi i \lambda_2)^{k-1} r^{2k-2}}\right) + O\left(\frac{1}{\lambda_2^{d+1} r^{2d+1}}\right).$$

Here the error is

$$(4.18) \quad \ll \frac{M^{2d+2}}{T^{d+1} r^{2d+1}}$$

by (3.55).

Substituting (4.17) and (4.18) into (4.15), we get for  $k = 2j$

$$\begin{aligned} & \int_{-r}^r e(\lambda_2 y^2) y^k dy \\ &= \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \frac{e(\frac{1}{8})}{\sqrt{f''(\gamma)}} \\ & \quad + 2e(\lambda_2 r^2) \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \frac{1}{4\pi i \lambda_2 r} \left(1 + \sum_{k=2}^d \frac{(2k-3)!!}{(4\pi i \lambda_2)^{k-1} r^{2k-2}}\right) \\ & \quad + \left[ e(\lambda_2 y^2) \cdot \sum_{i=1}^j \phi_i^{(k)}(y) \right]_{-r}^r + O\left(\frac{M^{2d+2j+2}}{T^{d+j+1} r^{2d+1}}\right). \end{aligned}$$

By (4.11) and (4.12) we see that the second term is exactly equal to  $\sum_{i=j+1}^{j+d} \phi_i^{(k)}(r)$ . Therefore

$$(4.19) \quad \begin{aligned} \int_{-r}^r e(\lambda_2 y^2) y^k dy &= \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \frac{e(\frac{1}{8})}{\sqrt{f''(\gamma)}} + \left[ e(\lambda_2 y^2) \cdot \sum_{i=1}^{j+d} \phi_i^{(k)}(y) \right]_{-r}^r \\ & \quad + O\left(\frac{M^{2d+2j+2}}{T^{d+j+1} r^{2d+1}}\right). \end{aligned}$$

For  $j \leq n+1$ , we take  $d = n+1-j$  in (4.19). Then by (4.6) and (4.19) we have

$$(4.20) \quad \begin{aligned} & \sum_{k=0}^{2n} \varpi_k \int_{-r}^r e(\lambda_2 y^2) y^k dy \\ &= \frac{e(\frac{1}{8})}{\sqrt{f''(\gamma)}} \sum_{j=0}^n \varpi_{2j} \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} + e(\lambda_2 r^2) \left[ \sum_{k=0}^{2n} \varpi_k \sum_{i=1}^{n+1} \phi_i^{(k)}(y) \right]_{-r}^r \\ & \quad + O\left(\sum_{j=0}^n U\left(\frac{M}{N} + 1\right)^{2j} \frac{M^{2n-2j+4}}{T^{n+2} r^{2n-2j+3}}\right), \end{aligned}$$

where we add the terms of odd  $k$ , which do not affect last equation by (4.14).

Now we estimate the second integral on the right hand side of (4.5). We have

$$(4.21) \quad Q(y) = g(x) \frac{dx}{dy} - \sum_{k=0}^{2n} \varpi_k y^k.$$

Therefore from (3.35) we get

$$(4.22) \quad Q(y) \ll_n U|y|^{2n+1} \left( \frac{1}{N^{2n+1}} + \frac{1}{M^{2n+1}} \right).$$

Similarly by (4.21) and (3.46) we see that for  $1 \leq t \leq n+2$

$$Q^{(t)}(y) = \frac{d^t}{dy^t} \left( g(x) \frac{dx}{dy} \right) - \sum_{k=0}^{2n-t} \frac{(k+t)!}{k!} \varpi_{k+t} y^k.$$

Therefore by (3.46) for  $1 \leq t \leq n+2$

$$(4.23) \quad Q^{(t)}(y) \ll U|y|^{2n+1-t} \left( \frac{1}{N^{2n+1}} + \frac{1}{M^{2n+1}} \right).$$

Next we choose a real number  $\delta \asymp \frac{1}{\sqrt{\lambda_2}}$  such that  $\frac{r}{\delta}$  is a power of 2. We have

$$(4.24) \quad V(Q(y)) \ll \int_{-\delta}^{\delta} \left| \frac{dQ}{dy} \right| dy \ll U|\delta|^{2n+1} \left( \frac{1}{N^{2n+1}} + \frac{1}{M^{2n+1}} \right).$$

From (4.21), (4.22), (4.24), and by the Second Derivative Test (see Lemma 5.1.3 of [10], P.88) we have

$$(4.25) \quad \begin{aligned} \int_{-\delta}^{\delta} Q(y) e(\lambda_2 y^2) dy &\ll \frac{\max_{-\delta \leq y \leq \delta} |Q(y)| + V(Q(y))}{\sqrt{\lambda_2}} \\ &\ll \frac{U|\delta|^{2n+1}}{\sqrt{\lambda_2}} \left( \frac{1}{N^{2n+1}} + \frac{1}{M^{2n+1}} \right) \\ &\ll \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right). \end{aligned}$$

Note that we use

$$\delta \asymp \frac{1}{\sqrt{\lambda_2}}, \quad \lambda_2 \gg \frac{T}{M^2}$$

in (4.25).

We split the range  $\delta \leq y \leq r, -r \leq y \leq -\delta$  into intervals of the form  $t \leq y \leq 2t, -2t \leq y \leq -t$ . By integration by parts we have

$$(4.26) \quad \int_t^{2t} Q(y) e(\lambda_2 y^2) dy = \left[ e(\lambda_2 y^2) \cdot \sum_{i=1}^{n+1} \psi_i(y) \right]_t^{2t} - \int_t^{2t} e(\lambda_2 y^2) \psi'_{n+1}(y) dy.$$

where

$$(4.27) \quad \psi_1(y) = \frac{Q(y)}{(4\pi i \lambda_2) y} \text{ and } \psi_k(y) = -\frac{\psi'_{k-1}(y)}{(4\pi i \lambda_2) y} \text{ for } 2 \leq k \leq n+3.$$

From (4.23) we see that for  $1 \leq k \leq n+3$

$$(4.28) \quad \begin{aligned} \psi_k(y) &= \sum_{i=0}^{k-1} c_{ki} \frac{Q^{(i)}(y)}{(4\pi i \lambda_2)^k y^{2k-1-i}} \ll \frac{U|y|^{2n+2-2k}}{\lambda_2^k} \left( \frac{1}{M^{2n+1}} + \frac{1}{N^{2n+1}} \right) \\ &\ll \frac{U t^{2n+2-2k}}{\lambda_2^k} \left( \frac{1}{M^{2n+1}} + \frac{1}{N^{2n+1}} \right), \end{aligned}$$

since  $t \leq |y| \leq 2t$ . Hence

$$\psi'_{n+1}(y) = -4\pi i \lambda_2 y \psi_{n+2}(y) \ll \frac{U}{\lambda_2^{n+1} t} \left( \frac{1}{M^{2n+1}} + \frac{1}{N^{2n+1}} \right),$$

and the total variation of  $\psi'_{n+1}(y)$  on  $[t, 2t]$  is

$$\begin{aligned} V(\psi'_{n+1}(y)) &= \int_t^{2t} |\psi'_{n+1}(y)| dy = \int_t^{2t} | -4\pi i \lambda_2 y \psi_{n+2}(y) + (4\pi i \lambda_2 y)^2 \psi_{n+3}(y) | dy \\ &\ll \frac{U}{\lambda_2^{n+1} t} \left( \frac{1}{M^{2n+1}} + \frac{1}{N^{2n+1}} \right). \end{aligned}$$

By the First Derivative Test (Lemma 5.1.2 of [10], P.88) we get

$$\begin{aligned} \int_t^{2t} e(\lambda_2 y^2) \psi'_n(y) dy &\ll \frac{\max_{t \leq y \leq 2t} \{\psi'_n(y)\} + V(\psi'_n(y))}{\lambda_2 t} \\ &\ll \frac{U}{\lambda_2^{n+2} t^2} \left( \frac{1}{N^{2n+1}} + \frac{1}{M^{2n+1}} \right). \end{aligned}$$

Note that  $\lambda_2 \gg \frac{T}{M^2}$  and by (4.26) we get

$$\int_t^{2t} Q(y) e(\lambda_2 y^2) dy = \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^{n+1} \psi_k(y) \right]_t^{2t} + O\left( \frac{UM^2}{T^{n+2} t^2} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).$$

Summing over ranges with  $t = 2^k \delta, k = 0, 1, 2, \dots$ , we get

$$\begin{aligned} (4.29) \quad \int_\delta^r Q(y) e(\lambda_2 y^2) dy &= \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^{n+1} \psi_k(y) \right]_\delta^r + O\left( \frac{UM^2}{T^{n+2} \delta^2} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \sum_{k \geq 1} \frac{1}{2^{2k}} \right) \\ &= \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^n \psi_k(y) \right]_\delta^r + O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right). \end{aligned}$$

Similarly

$$(4.30) \quad \int_{-r}^{-\delta} Q(y) e(\lambda_2 y^2) dy = \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^n \psi_k(y) \right]_{-r}^{-\delta} + O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).$$

Now by (4.29), (4.30) and (4.25) we see that

$$\begin{aligned} \int_{-r}^r Q(y) e(\lambda_2 y^2) dy &= \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^n \psi_k(y) \right]_{-r}^r - \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^n \psi_k(y) \right]_{-\delta}^\delta \\ &\quad + O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right). \end{aligned}$$

By (4.28) we have the following trivial estimates

$$\begin{aligned} \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^{n+1} \psi_k(y) \right]_{-\delta}^\delta &\ll \sum_{k=1}^{n+1} \frac{U \delta^{2n+2-2k}}{\lambda_2^k} \left( \frac{1}{M^{2n+1}} + \frac{1}{N^{2n+1}} \right) \\ &\ll \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right). \end{aligned}$$

Therefore

$$(4.31) \quad \int_{-r}^r Q(y) e(\lambda_2 y^2) dy = \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^{n+1} \psi_k(y) \right]_{-r}^r + O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).$$



Now by (4.5), (4.20), (4.31) we have

$$\begin{aligned}
(4.32) \quad \int_u^v g(x)e(f(x))dx &= e(f(\gamma)) \int_{-r}^r e(\lambda_2 y^2) g(x) \frac{dx}{dy} dy \\
&= e(f(\gamma)) \frac{e(\frac{1}{8})}{\sqrt{f''(\gamma)}} \sum_{j=0}^n \varpi_{2j} \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \\
&\quad + e(f(\gamma)) \left[ e(\lambda_2 y^2) \sum_{k=0}^{2n} \varpi_k \sum_{i=1}^{n+1} \phi_i^{(k)}(y) \right]_{-r}^r \\
&\quad + e(f(\gamma)) \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^{n+1} \psi_k(y) \right]_{-r}^r \\
&\quad + O\left( \sum_{j=0}^n U \left( \frac{M}{N} + 1 \right)^{2j} \frac{M^{2n-2j+4}}{T^{n+2} r^{2n-2j+3}} \right) \\
&\quad + O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).
\end{aligned}$$

Choosing  $r$  in the range

$$(4.33) \quad \frac{M}{T^{\frac{1}{2n+3}}} < r < M,$$

which is possible if  $T$  is sufficiently large, then the first  $O$ -term in (4.32) is

$$\begin{aligned}
&\ll U \left( \frac{M}{N} + 1 \right)^{2n} \frac{1}{T^{n+2}} \sum_{j=0}^n \frac{M^{2n-2j+4}}{r^{2n-2j+3}} \\
&\ll U \left( \frac{M^{2n}}{N^{2n}} + 1 \right) \frac{1}{T^{n+2}} \frac{M^{2n+4}}{r^{2n+3}} \\
&\ll U \frac{1}{T^{n+1}} \left( \frac{M^{2n+1}}{N^{2n}} + M \right).
\end{aligned}$$

Therefore by (4.32)

$$\begin{aligned}
(4.34) \quad \int_u^v g(x)e(f(x))dx &= e(f(\gamma)) \frac{e(\frac{1}{8})}{\sqrt{f''(\gamma)}} \sum_{j=0}^n \varpi_{2j} \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \\
&\quad + e(f(\gamma)) \left[ e(\lambda_2 y^2) \sum_{k=0}^{2n} \varpi_k \sum_{i=1}^{n+1} \phi_i^{(k)}(y) \right]_{-r}^r \\
&\quad + e(f(\gamma)) \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^{n+1} \psi_k(y) \right]_{-r}^r \\
&\quad + O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).
\end{aligned}$$

Let

$$\begin{aligned}
F(y) &= f(\gamma) + \lambda_2 y^2 = f(x), \quad G(y) = g(x) \frac{dx}{dy}, \\
\theta_1(y) &= \frac{G(y)}{2\pi i F'(y)}, \quad \theta_i(y) = -\frac{\theta'_{i-1}(y)}{2\pi i F'(y)} \text{ for } 2 \leq i \leq n+1.
\end{aligned}$$

Now we want to show the following two equalities,

$$\begin{aligned}
(4.35) \quad \left[ e(F(y)) \cdot \sum_{i=1}^{n+1} \theta_i(y) \right]_{-r}^r &= e(f(\gamma)) \left[ e(\lambda_2 y^2) \sum_{k=0}^{2n} \varpi_k \sum_{i=1}^{n+1} \phi_i^{(k)}(y) \right]_{-r}^r \\
&\quad + e(f(\gamma)) \left[ e(\lambda_2 y^2) \cdot \sum_{k=1}^{n+1} \psi_k(y) \right]_{-r}^r
\end{aligned}$$

and

$$(4.36) \quad \left[ e(F(y)) \cdot \sum_{i=1}^{n+1} \theta_i(y) \right]_{-r}^r = \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_u^v$$

where  $H_i(x)$  is defined in (2.4). By (4.21) we see that

$$G(y) = \sum_{k=0}^{2n} \varpi_k y^k + Q(y),$$

and then

$$\theta_1(y) = \sum_{k=0}^{2n} \varpi_k \frac{y^{k-1}}{4\pi i \lambda_2} + \frac{Q(y)}{4\pi i \lambda_2 y}.$$

By (4.8) and (4.27) we get

$$\theta_1(y) = \sum_{k=0}^{2n} \varpi_k \phi_1^{(k)}(y) + \psi_1(y),$$

and again by (4.8), (4.9) and (4.27) we see that for  $1 \leq i \leq n+1$

$$\theta_i(y) = \sum_{k=0}^{2n} \varpi_k \phi_i^{(k)}(y) + \psi_i(y).$$

Therefore (4.35) is true.

To prove (4.36) we see that

$$F'(y) = f'(x) \frac{dx}{dy} \text{ and } G(y) = g(x) \frac{dx}{dy},$$

and hence

$$\theta_1(y) = \frac{G(y)}{2\pi i F'(y)} = \frac{g(x)}{2\pi i f'(x)} = H_1(x),$$

$$\theta_i(y) = -\frac{\theta'_{i-1}(y)}{2\pi i F'(y)} = -\frac{H'_{i-1}(x)}{2\pi i f'(x)} = H_i(x) \text{ for } 2 \leq i \leq n+1.$$

Now by induction (4.36) follows from the last two formulas and the correspondence between  $y = r$  and  $x = v$ , and between  $y = -r$  and  $x = u$ .

Then we can conclude from (4.34), (4.35) and (4.36)

$$(4.37) \quad \int_u^v g(x) e(f(x)) dx = \frac{e\left(f(\gamma) + \frac{1}{8}\right)}{\sqrt{f''(\gamma)}} \left( g(\gamma) + \sum_{j=1}^n \varpi_{2j} \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \right) \\ + \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_u^v + O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).$$

By

$$\frac{M}{T^{\frac{1}{2n+3}}} \leq r \leq M$$

we get that the  $O$ -terms in (4.1) are

$$\begin{aligned}
&\ll \frac{UM^{2n+4}}{T^{n+2}r^{n+2}} \left[ \frac{1}{N^{n+1}} \left( \frac{M}{N} + 1 \right) + \sum_{j=1}^{n+1} \frac{1}{r^j} \sum_{t=0}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right] \\
&\ll \frac{UM^{2n+4}}{T^{n+2}r^{n+2}} \left[ \frac{1}{N^{n+1}} \left( \frac{M}{N} + 1 \right) + \sum_{j=1}^{n+1} \frac{1}{r^j} \left( \frac{1}{N^{n+1-j}} + \frac{1}{M^{n+1-j}} \right) \right] \\
&\ll \frac{UM^{n+2}T^{\frac{n+2}{2n+3}}}{T^{n+2}} \left[ \frac{1}{N^{n+1}} \left( \frac{M}{N} + 1 \right) + \sum_{j=1}^{n+1} T^{\frac{j}{2n+3}} \left( \frac{1}{N^{n+1-j} M^j} + \frac{1}{M^{n+1}} \right) \right] \\
&\ll \frac{UM^{n+2}T^{\frac{n+2}{2n+3}}}{T^{n+2}} \left[ \frac{1}{N^{n+1}} \left( \frac{M}{N} + 1 \right) + T^{\frac{n+1}{2n+3}} \left( \frac{1}{M^{n+1}} + \frac{1}{N^{n+1}} \right) \right] \\
&\ll \frac{UM^{n+2}T^{\frac{n+2}{2n+3} + \frac{n+1}{2n+3}}}{T^{n+2}} \left[ \frac{M}{N^{n+2}} + \frac{1}{N^{n+1}} + \frac{1}{M^{n+1}} \right] \\
&\ll \frac{U}{T^{n+1}} \left[ \frac{M^{n+3}}{N^{n+2}} + \frac{M^{n+2}}{N^{n+1}} + M \right] \\
&\ll \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right)
\end{aligned}$$

and

$$\begin{aligned}
&\ll \frac{UM^{2n+5}}{NT^{n+2}r^{n+12}} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \frac{1}{r^j} \sum_{t=j}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \\
&\ll \frac{UM^{n+3}T^{\frac{n+2}{2n+3}}}{NT^{n+2}} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} T^{\frac{j}{2n+1}} \left[ \frac{1}{N^{n+1-2j} M^{2j}} + \frac{1}{M^{n+1}} \right] \\
&\ll \frac{UM^{n+3}T^{\frac{n+2}{2n+3}}}{NT^{n+2}} T^{\frac{\lfloor \frac{n+1}{2} \rfloor}{2n+3}} \left( \frac{1}{M^{n+1}} + \frac{1}{M^2 N^{n-1}} \right) \\
&\ll \frac{U}{T^{n+1}} \left( \frac{M^{n+1}}{N^n} + \frac{M^2}{N} \right) \\
&\ll \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right).
\end{aligned}$$

Hence for  $\alpha \leq u$

$$(4.38) \quad \int_{\alpha}^u g(x) e(f(x)) dx = \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_{\alpha}^u + O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).$$

Similarly for  $v \leq \beta$

$$(4.39) \quad \int_v^{\beta} g(x) e(f(x)) dx = \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_v^{\beta} + O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).$$

At last, we consider the whole integral

$$\int_{\alpha}^{\beta} g(x) e(f(x)) dx.$$

Since

$$\int_{\alpha}^{\beta} = \int_{\alpha}^u + \int_u^v + \int_v^{\beta} \quad \left( \text{or } \int_u^v - \int_u^{\alpha} - \int_{\beta}^v \right),$$

then by (4.2), (4.4), (4.37), (4.38), (4.39) we conclude

$$\begin{aligned}
& \int_{\alpha}^{\beta} g(x)e(f(x))dx \\
&= \frac{e\left(f(\gamma) + \frac{1}{8}\right)}{\sqrt{f''(\gamma)}} \left( g(\gamma) + \sum_{j=1}^n \varpi_{2j} \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \right) + \left[ e(f(x)) \cdot \sum_{i=1}^{n+1} H_i(x) \right]_{\alpha}^{\beta} \\
&+ O\left( \frac{UM^{2n+5}}{NT^{n+2}} \sum_{j=1}^{\lfloor \frac{n+1}{2} \rfloor} \left( \frac{1}{(\gamma - \alpha)^{n+2+j}} + \frac{1}{(\beta - \gamma)^{n+2+j}} \right) \sum_{t=j}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right) \\
&+ O\left( \frac{UM^{2n+4}}{T^{n+2} N^{n+1}} \left( \frac{M}{N} + 1 \right) \left( \frac{1}{(\gamma - \alpha)^{n+2}} + \frac{1}{(\beta - \gamma)^{n+2}} \right) \right) \\
&+ O\left( \frac{UM^{2n+4}}{T^{n+2}} \sum_{j=1}^{n+1} \left( \frac{1}{(\gamma - \alpha)^{n+2+j}} + \frac{1}{(\beta - \gamma)^{n+2+j}} \right) \sum_{t=0}^{n+1-j} \frac{1}{N^{n+1-j-t} M^t} \right) \\
&+ O\left( \frac{U}{T^{n+1}} \left( \frac{M^{2n+2}}{N^{2n+1}} + M \right) \right).
\end{aligned}$$

□

## 5. BACKGROUND ON AUTOMORPHIC FORMS

We will follow the setting and notations in Li [19]. Recall for  $m, n \geq 1$  the Kuznetsov trace formula (Kuznetsov [15] and Conrey and Iwaniec [4])

$$\begin{aligned}
(5.1) \quad & \sum'_{j \geq 1} h(t_j) \omega_j \lambda_j(m) \lambda_j(n) + \frac{1}{4\pi} \int_{\mathbb{R}} h(t) \omega(t) \bar{\eta} \left( m, \frac{1}{2} + it \right) \eta \left( n, \frac{1}{2} + it \right) dt \\
&= \delta(m, n) \frac{H}{2} + \sum_{c \geq 1} \frac{1}{2c} \left\{ S(m, n; c) H^+ \left( \frac{4\pi \sqrt{mn}}{c} \right) + S(-m, n; c) H^- \left( \frac{4\pi \sqrt{mn}}{c} \right) \right\}.
\end{aligned}$$

Here  $\sum'$  means we are only summing over even Maass forms,  $\delta(m, n)$  is the Kronecker delta,

$$\begin{aligned}
\omega_j &= \frac{4\pi |\rho_j(1)|^2}{\cosh \pi t_j}, \quad \omega(t) = 4\pi \frac{|\phi(1, 1/2 + it)|^2}{\cosh \pi t}, \\
H &= \frac{2}{\pi} \int_0^{\infty} h(t) \tanh(\pi t) t \, dt, \quad H^+(x) = 2i \int_{\mathbb{R}} J_{2it}(x) \frac{h(t)t}{\cosh \pi t} \, dt, \\
H^-(x) &= \frac{4}{\pi} \int_{\mathbb{R}} K_{2it}(x) \sinh(\pi t) h(t)t \, dt, \quad \text{and } S(a, b; c) = \sum_{d\bar{d} \equiv 1 \pmod{c}} e\left(\frac{da + \bar{d}b}{c}\right)
\end{aligned}$$

is the standard Kloosterman sum. Above,  $J_{\nu}$  is the  $J$ -Bessel function.

We let  $f$  be a Maass form of type  $\nu = (\nu_1, \nu_2)$  for  $SL_3(\mathbb{Z})$  (cf. Goldfeld [6]). Then  $f$  has a Whittaker function expansion

$$(5.2) \quad f(z) = \sum_{\pm \Gamma^{\infty} \backslash SL_2(\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1, m_2)}{m_1 |m_2|} W_J \left( M \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z, \nu, \psi_{1,1} \right)$$

where  $W_J$  is the Jacquet-Whittaker function,  $M = \text{diag}(m_1 |m_2|, m_1, 1)$ , and  $\psi_{1,1}$  is a fixed specific generic character on the abelianization of the standard unipotent upper triangular subgroup of  $SL_3(\mathbb{Z})$ . Put

$$\alpha = -\nu_1 - 2\nu_2 + 1, \quad \beta = -\nu_1 + \nu_2, \quad \gamma = 2\nu_1 + \nu_2 - 1.$$

These are the Langlands parameters at  $\infty$  of  $f$ . In the usual way, we put  $\tilde{\psi}(s) = \int_0^{\infty} \psi(x) x^{s-1} dx$  to be the Mellin transform of  $\psi$  which we assume is smooth and compactly supported on  $(0, \infty)$ .

For  $k = 0, 1$  we define

$$\Psi_k(x) = \int_{\text{Res}=\sigma} (\pi^3 x)^{-s} \frac{\Gamma(\frac{1+s+2k+\alpha}{2})\Gamma(\frac{1+s+2k+\beta}{2})\Gamma(\frac{1+s+2k+\gamma}{2})}{\Gamma(\frac{-s-\alpha}{2})\Gamma(\frac{-s-\beta}{2})\Gamma(\frac{-s-\gamma}{2})} \tilde{\psi}(-s-k) ds.$$

Here  $\sigma$  is taken sufficiently large depending on  $\alpha, \beta, \gamma$ . We then define, for  $k = 0, 1$ ,

$$\Psi_{0,1}^k(x) = \Psi_0(x) + (-1)^k \frac{1}{x\pi^{3i}} \Psi_1(x).$$

Then the following is a crucial tool, the Voronoi formula for  $GL(3)$ .

**Proposition 5.1.** ([23]) *Let  $\psi \in C_c^\infty(0, \infty)$ . Let  $f$  be a  $SL_3(\mathbb{Z})$  Maass form with corresponding Fourier coefficients  $A(m, n)$  as in (5.2). Let  $d, \bar{d}, c \in \mathbb{Z}$  with  $c \neq 0$ ,  $(d, c) = 1$ , and  $d\bar{d} \equiv 1 \pmod{c}$ . Then*

$$\begin{aligned} \sum_{n>0} A(m, n) e\left(\frac{n\bar{d}}{c}\right) \psi(n) &= \frac{c}{4\pi^{5/2}i} \sum_{n_1|cm} \sum_{n_2>0} \frac{A(n_2, n_1)}{n_1 n_2} S\left(md, n_2; \frac{mc}{n_1}\right) \Psi_{0,1}^0\left(\frac{n_1^2 n_2}{c^3 m}\right) \\ &+ \frac{c}{4\pi^{5/2}i} \sum_{n_1|cm} \sum_{n_2>0} \frac{A(n_2, n_1)}{n_1 n_2} S\left(md, -n_2; \frac{mc}{n_1}\right) \Psi_{0,1}^1\left(\frac{n_1^2 n_2}{c^3 m}\right). \end{aligned}$$

To use this formula, asymptotics of  $\Psi_0, \Psi_1$  are needed which were proved in Li [18] and Ren and Ye [25] for  $GL(3)$ . (For  $GL(m)$  see Ren and Ye [26].) Since  $x^{-1}\Psi_1(x)$  has similar asymptotics to  $\Psi_0$ , following [19], we only deal with  $\Psi_0$ . We will use the following Lemma ([18]):

**Lemma 5.2.** *Suppose  $\psi \in C_c^\infty([X, 2X])$ . Then for any fixed integer  $K \geq 1$  and  $xX \gg 1$  we have*

$$\Psi_0(x) = 2\pi^3 xi \int_0^\infty \psi(y) \sum_{j=1}^K \frac{c_j \cos(6\pi(xy)^{1/3}) + d_j \sin(6\pi(xy)^{1/3})}{(xy)^{1/3}} dy + O((xX)^{\frac{2-K}{3}})$$

Here  $c_j$  and  $d_j$  are constants depending on the Langlands parameters.  $c_1 = 0$  and  $d_1 = -2/\sqrt{3\pi}$ .

We now assume  $f$  is a self-dual Hecke-Maass form for  $SL_3(\mathbb{Z})$  of type  $(\nu, \nu)$ , normalized so that  $A(1, 1) = 1$ . The Rankin-Selberg  $L$ -function of  $f$  with itself is then defined by

$$L(s, f \times f) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{|A(m, n)|^2}{(m^2 n)^s}$$

for  $\text{Res}$  large.  $L(s, f \times f)$  has meromorphic continuation to the complex plane, with a simple pole at  $s = 1$ . By a standard analytic number theory argument using complex analysis, this gives

$$\sum_{m^2 n \leq N} |A(m, n)|^2 \ll_f N.$$

Applying Cauchy-Schwartz, this gives

$$(5.3) \quad \sum_{n \leq N} |A(m, n)| \ll_f |m| N.$$

We will use (5.3) and summation by parts in the estimates below. Here  $f$  being self-dual also means  $A(m, n) = A(n, m)$  for all  $m, n$ .

The Rankin-Selberg  $L$ -function of  $f$  with  $u_j$  is (for  $\text{Res}$  sufficiently large)

$$L(s, f \times u_j) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_j(n) A(m, n)}{(m^2 n)^s}.$$

$L(s, f \times u_j)$  can be completed to  $\Lambda(s, f \times u_j)$  with six  $\Gamma$  factors at  $\infty$  (involving the Langlands parameters of  $f$ , and  $t_j$ ).

We now need to define the Rankin-Selberg  $L$ -function of  $f$  with the Eisenstein series. See Li [19] for the definition of  $E(z, s)$  and  $\eta(n, s)$ .

$$L(s, f \times E) = \sum_{m \geq 1} \sum_{n \geq 1} \frac{\bar{\eta}(n, 1/2 + it) A(m, n)}{(m^2 n)^s}.$$

Following Goldfeld [6], comparing Euler products, we have

$$L\left(\frac{1}{2}, f \times E\right) = \left| L\left(\frac{1}{2} - it, f\right) \right|^2.$$

We need to set up the approximate functional equation. We define

$$\begin{aligned} \gamma(s, t) &= \pi^{-3s} \Gamma\left(\frac{s - it - \alpha}{2}\right) \Gamma\left(\frac{s - it - \beta}{2}\right) \Gamma\left(\frac{s - it - \gamma}{2}\right) \\ &\quad \times \Gamma\left(\frac{s + it - \alpha}{2}\right) \Gamma\left(\frac{s + it - \beta}{2}\right) \Gamma\left(\frac{s + it - \gamma}{2}\right). \end{aligned}$$

Here  $\alpha = -3\nu + 1$ ,  $\beta = 0$ , and  $\gamma = 3\nu - 1$  are the Langlands parameters at  $\infty$  of  $f$ . We define

$$F(u) = \left( \cos\left(\frac{\pi u}{A}\right) \right)^{-3A}$$

for  $A$  a positive integer. For  $|\text{Im}t| \leq 1000$  we now define

$$(5.4) \quad V(y, t) = \frac{1}{2\pi i} \int_{(1000)} y^{-u} F(u) \frac{\gamma(1/2 + u, t)}{\gamma(1/2, t)} \frac{du}{u}.$$

By known bounds for the Langlands parameters, this integral converges. We have the important approximate functional equation (cf. [19]):

**Lemma 5.3.** *For  $f$  a self-dual Maass form of type  $(\nu, \nu)$  for  $SL_3(\mathbb{Z})$  and  $u_j$  a Hecke-Maass form for  $SL_2(\mathbb{Z})$  corresponding to the eigenvalue  $1/4 + t_j^2$  in an orthonormal basis, as above,*

$$(5.5) \quad L\left(\frac{1}{2}, f \times u_j\right) = 2 \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_j(n) A(m, n)}{\sqrt{m^2 n}} V(m^2 n, t_j).$$

The point of using  $V$  in the expansion (5.5) is that  $V$  decays rapidly for  $m^2 n \gg |t_j|^{3+\varepsilon}$ , and so in an effective way, we can take both sums above to be finite. For the precise decay rate, see Lemma 2.3 of Li [19].

We also have the approximate functional equation for  $L(s, f \times E)$ :

$$(5.6) \quad L\left(\frac{1}{2}, f \times E\right) = 2 \sum_{m \geq 1} \sum_{n \geq 1} \frac{\eta(n, 1/2 + it) A(m, n)}{\sqrt{m^2 n}} V(m^2 n, t).$$

Following Li [19] we now define

$$W = \sum_j' e^{-(\frac{t_j - T}{M})^2} \omega_j L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{\mathbb{R}} e^{-(\frac{t - T}{M})^2} \omega(t) \left| L\left(\frac{1}{2} - it, f\right) \right|^2 dt.$$

Here  $\omega_j$  and  $\omega(t)$  are defined below (5.1). It is known that  $\omega_j \gg t_j^{-\varepsilon}$  and  $\omega(t) \gg t^{-\varepsilon}$ . (See the references in Li [19]) It follows that

$$\sum_j' e^{-(\frac{t_j - T}{M})^2} L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{\mathbb{R}} e^{-(\frac{t - T}{M})^2} \left| L\left(\frac{1}{2} - it, f\right) \right|^2 dt \ll WT^\varepsilon.$$

Consequently, our main Theorem will be proved if we show  $W \ll_{\varepsilon, f} T^{1+\varepsilon} M$ . As Li [19] points out, the function  $e^{-(\frac{t - T}{M})^2}$  cannot be used as a test function in the Kuznetsov trace formula simply because it is not even. Following Li [19] we will use the modified function

$$(5.7) \quad k(t) = e^{-(\frac{t - T}{M})^2} + e^{-(\frac{t + T}{M})^2}$$

which essentially captures the size of  $e^{-(\frac{t-T}{M})^2}$  for  $t$  near  $T$ . Thus, we define

$$(5.8) \quad \mathcal{W} = \sum_j' k(t_j) \omega_j L\left(\frac{1}{2}, f \times u_j\right) + \frac{1}{4\pi} \int_{\mathbb{R}} k(t) \omega(t) \left| L\left(\frac{1}{2} - it, f\right) \right|^2 dt.$$

By plugging (5.5) and (5.6) into  $\mathcal{W}$  in (5.8) we see that we need to analyze  $\mathcal{R}$  which we define by the equation

$$(5.9) \quad \begin{aligned} \mathcal{R} = & 2 \sum_j' k(t_j) \omega_j \sum_{m \geq 1} \sum_{n \geq 1} \frac{\lambda_j(n) A(m, n)}{\sqrt{m^2 n}} V(m^2 n, t_j) g\left(\frac{m^2 n}{N}\right) \\ & + \frac{1}{2\pi} \int_{\mathbb{R}} k(t) \omega(t) \sum_{m \geq 1} \sum_{n \geq 1} \frac{\eta(n, 1/2 + it) A(m, n)}{\sqrt{m^2 n}} V(m^2 n, t) g\left(\frac{m^2 n}{N}\right) dt. \end{aligned}$$

Here, for the rest of this article we take  $N = T^{3+\varepsilon}$  and  $g$  is a fixed non-negative function with compact support in  $[1, 2]$ . This is the trick of using a dyadic partition of unity which is best outlined in Lau, Liu, and Ye [17].

Now, we apply the Kuznetsov trace formula (5.1) to  $\mathcal{R}$  (5.9). Consequently, we write

$$(5.10) \quad \begin{aligned} \mathcal{R} &= \mathcal{D} + \mathcal{R}^+ + \mathcal{R}^-; \\ \mathcal{D} &= \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{\sqrt{m^2 n}} g\left(\frac{m^2 n}{N}\right) \delta(n, 1) H_{m, n}; \\ H_{m, n} &= \frac{2}{\pi} \int_{\mathbb{R}} k(t) V(m^2 n, t) \tanh(\pi t) t dt; \\ \mathcal{R}^+ &= \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{\sqrt{m^2 n}} g\left(\frac{m^2 n}{N}\right) \sum_{c > 0} \frac{S(n, 1; c)}{c} H_{m, n}^+\left(\frac{4\pi\sqrt{n}}{c}\right); \end{aligned}$$

$$(5.11) \quad \begin{aligned} H_{m, n}^+(x) &= 2i \int_{\mathbb{R}} J_{2it}(x) \frac{k(t) V(m^2 n, t) t}{\cosh(\pi t)} dt; \\ \mathcal{R}^- &= \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{\sqrt{m^2 n}} g\left(\frac{m^2 n}{N}\right) \sum_{c > 0} \frac{S(n, -1; c)}{c} H_{m, n}^-\left(\frac{4\pi\sqrt{n}}{c}\right); \\ H_{m, n}^-(x) &= \frac{4}{\pi} \int_{\mathbb{R}} K_{2it}(x) \sinh(\pi t) k(t) V(m^2 n, t) t dt. \end{aligned}$$

By the estimates in Section 3 of Li [19], we see easily that  $\mathcal{D}$  is negligible for any  $M$  with  $T^\varepsilon \leq M \leq T^{1-\varepsilon}$  and we leave the details for the reader. In the next two section we will estimate  $\mathcal{R}^+$  and  $\mathcal{R}^-$ .

## 6. ESTIMATES FOR THE $J$ -BESSEL FUNCTION TERMS

In this section we provide estimates for  $\mathcal{R}^+$  in (5.10). In this section and the next, we show estimates under the assumption  $T^{1/3+2\varepsilon} \leq M \leq T^{1/2}$ . Following Li [19] we define the parameters

$$(6.1) \quad C_1 = T^{100}, \text{ and } C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon} M},$$

and we split  $\mathcal{R}^+ = \mathcal{R}_1^+ + \mathcal{R}_2^+ + \mathcal{R}_3^+$  with

$$(6.2) \quad \mathcal{R}_1^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{\sqrt{m^2 n}} g\left(\frac{m^2 n}{N}\right) \sum_{c \geq C_1/m} \frac{S(n, 1; c)}{c} H_{m, n}^+\left(\frac{4\pi\sqrt{n}}{c}\right),$$

$$(6.3) \quad \mathcal{R}_2^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{\sqrt{m^2 n}} g\left(\frac{m^2 n}{N}\right) \sum_{C_2/m \leq c \leq C_1/m} \frac{S(n, 1; c)}{c} H_{m, n}^+\left(\frac{4\pi\sqrt{n}}{c}\right),$$

and

$$(6.4) \quad \mathcal{R}_3^+ = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{\sqrt{m^2 n}} g\left(\frac{m^2 n}{N}\right) \sum_{c \leq C_2/m} \frac{S(n, 1; c)}{c} H_{m, n}^+\left(\frac{4\pi\sqrt{n}}{c}\right).$$

For  $\mathcal{R}_1^+$ , Li [19] shifts the integral defining  $H_{m, n}^+$  (see (5.11)), and uses an integral representation of the  $J$ -Bessel function and Stirling's formula to conclude

$$(6.5) \quad H_{m, n}^+(x) \ll x^{\frac{3}{4}} T^{\frac{3}{8}} (m^2 n)^{-\frac{3}{8}} T^{1+\varepsilon} M.$$

Consequently

$$\mathcal{R}_1^+ \ll T^{\frac{11}{8}+\varepsilon} M \sum_{m \leq \sqrt{2N}} \sum_{n \leq 2N/m^2} \frac{|A(m, n)|}{m\sqrt{n}} \sum_{c \geq C_1/m} \frac{|S(n, 1; c)|}{c} \left(\frac{\sqrt{n}}{c}\right)^{\frac{3}{4}} \cdot (m^2 n)^{-\frac{3}{8}}.$$

Using Weil's bound for  $S(n, 1; c)$ , we see

$$\sum_{c \geq C_1/m} \frac{|S(n, 1; c)|}{c^{\frac{7}{4}}} \ll \sum_{c \geq C_1/m} \frac{c^{\frac{1}{2}+\varepsilon}}{c^{\frac{7}{4}}} \ll \left(\frac{C_1}{m}\right)^{-\frac{1}{4}+\varepsilon}.$$

Using (5.3) and summation by parts, we see

$$\sum_{n \leq 2N/m^2} \frac{|A(m, n)|}{\sqrt{n}} \ll m \left(\frac{N}{m^2}\right)^{\frac{1}{2}}.$$

Using these last two estimates we see

$$\mathcal{R}_1^+ \ll T^{\frac{11}{8}+\varepsilon} M N^{\frac{1}{2}} C_1^{-\frac{1}{4}} \sum_{m \leq \sqrt{2N}} \frac{1}{m^{\frac{3}{2}}}.$$

Plugging in  $C_1 = T^{100}$ ,  $N = T^{3+\varepsilon}$  and noticing the sum on  $m$  converges, we have  $\mathcal{R}_1^+ \ll 1$  for any  $M$  with  $T^\varepsilon \leq M \leq T^{1-\varepsilon}$ .

We now deal with  $\mathcal{R}_2^+$ . We do not wish to reproduce all the estimates in Li [19] so we will summarize. As used in Liu and Ye [20] and Li [19] we need an integral representation for

$$\frac{J_{2it}(x) - J_{-2it}(x)}{\cosh(\pi t)}$$

from 1.13(69) of [2], vol.1, p.59. Using integration by parts, a change of variables, and the fact that  $k(t)$  (recall (5.7)) is a Schwartz function, we define

$$W_{m, n}(x) = T \int_{\mathbb{R}} \widehat{k^*}(\zeta) \cos\left(x \cosh\left(\frac{\zeta\pi}{M}\right)\right) e\left(-\frac{T\zeta}{M}\right) d\zeta.$$

Here

$$k^*(t) = e^{-t^2} V(m^2 n, tM + T)$$

is a Schwartz function, and  $\widehat{k^*}$  is its Fourier transform. We remark that derivatives of  $k^*(t)$  are  $\ll 1$ . In fact, by (5.4)  $\frac{\partial^\ell}{\partial t^\ell} V(y, tM + T)$  can be expressed in terms of derivatives of  $\gamma(s, tM + T)$  and hence in terms of  $\frac{d}{dz} \log \Gamma(z) =: \psi(z)$  and  $\psi^{(\ell)}(z)$  (Bateman [1] p.15, 1.7(1), and p.45, 1.16(9)). By their asymptotic expansions in [1], p.47, 1.18(7), and p.48, 1.18(9), we can see

$$\frac{\partial^\ell}{\partial t^\ell} V(y, tM + T) \ll \left(\frac{M}{T}\right)^\ell.$$

We define

$$(6.6) \quad W_{m, n}^*(x) = T \int_{\mathbb{R}} \widehat{k^*}(\zeta) e\left(-\frac{T\zeta}{M} - \frac{x}{2\pi} \cosh\left(\frac{\zeta\pi}{M}\right)\right) d\zeta,$$



so that

$$W_{m,n}(x) = \frac{W_{m,n}^*(x) + W_{m,n}^*(-x)}{2}.$$

The upshot here is that up to a lower order term (which can be handled in a similar way) and a negligible amount, we have

$$H_{m,n}^+(x) = 4W_{m,n}(x).$$

The contribution to the integral in (6.6) from  $|\zeta| \geq T^\varepsilon$  is a negligible amount, so in what follows we can assume  $|\zeta| \leq T^\varepsilon$ . The phase  $\phi(\zeta)$  in the exponential (6.6) is

$$2\pi \cdot \phi(\zeta) = -\frac{T\zeta}{M} - \frac{x}{2\pi} \cosh\left(\frac{\zeta\pi}{M}\right).$$

Looking at  $\phi'(\zeta)$ , we see  $W_{m,n}^*(x)$  is negligible for  $|x| \leq T^{1-\varepsilon}M$ . So in what follows we assume

$$T^{1-\varepsilon}M \leq |x| \leq T^2.$$

Using a Taylor expansion in  $\zeta$  (within the exponential) of

$$e\left(-\frac{T\zeta}{M} - \frac{x}{2\pi} \cosh\left(\frac{\zeta\pi}{M}\right)\right)$$

in (6.6), using the Fourier transform of a Gaussian, using Parseval's Theorem, completing the square, and working out many estimates Lau, Liu and Ye (Lemma 5.1 of [17]) and Li (Proposition 4.1 of [19]) proved similar propositions, estimating  $W_{m,n}^*(x)$  by a finite series involving derivatives of  $\widehat{k^*}$ , based on ideas in Sarnak [27]. For our purposes we can modify the proof of Proposition 4.1 of [19].

We have

**Proposition 6.1.** 1) For  $|x| \leq T^{1-\varepsilon}M$ ,

$$W_{m,n}^*(x) \ll_{\varepsilon,A} T^{-A}.$$

2) For  $T^{1-\varepsilon}M \leq |x| \leq T^2$ , with  $T^{1/3+2\varepsilon} \leq M \leq T^{1/2}$  and  $L_1, L_2 \geq 1$ ,

$$(6.7) \quad \begin{aligned} W_{m,n}^*(x) &= \frac{TM}{\sqrt{|x|}} e\left(-\frac{x}{2\pi} + \frac{T^2}{\pi x}\right) \sum_{l=0}^{L_1} \sum_{0 \leq l_1 \leq 2l} \sum_{\frac{l_1}{4} \leq l_2 \leq L_2} c_{l,l_1,l_2} \frac{M^{2l-l_1} T^{4l_2-l_1}}{x^{l+3l_2-l_1}} \\ &\quad \times \left[ \widehat{k^*}^{(2l-l_1)}\left(-\frac{2MT}{\pi x}\right) - \frac{\pi^6 ix}{6!M^6} (y^6 \widehat{k^*}(y))^{(2l-l_1)} \right. \\ &\quad \left. + \frac{\pi^{12} i^2 x^2}{2!(6!)^2 M^{12}} (y^{12} \widehat{k^*}(y))^{(2l-l_1)} \left(-\frac{2MT}{\pi x}\right) \right] \\ &\quad + O\left(\frac{TM}{\sqrt{|x|}} \left(\frac{T^4}{|x|^3}\right)^{L_2+1} + T \left(\frac{M}{\sqrt{|x|}}\right)^{2L_1+3} + \frac{T|x|^3}{M^{18}}\right), \end{aligned}$$

where  $c_{l,l_1,l_2}$  are constants depending only on the indices.

Note that part 1) is valid for  $T^\varepsilon \leq M \leq T^{1-\varepsilon}$ , and part 2) is valid for  $T^{1/3+\varepsilon} \leq M \leq \sqrt{T}$  with the assumption  $T^{1-\varepsilon}M \leq |x| \leq T^2$ . With our assumption on  $M$ , that  $T^{1/3+2\varepsilon} \leq M \leq \sqrt{T}$ , to acquire the desired decay rate of the

$$O\left(\frac{TM}{\sqrt{|x|}} \left(\frac{T^4}{|x|^3}\right)^{L_2+1}\right)$$

term,  $L_2$  could depend on  $\varepsilon$ . From 1) of Proposition 6.1 we see  $\mathcal{R}_2^+$  is negligible. The extra term in the brackets in (6.7), as compared to [19]), comes from a degree 2 Taylor expansion in  $x$  (with remainder) of

$$e\left(-\frac{\pi^6 ix \zeta^6}{2 \cdot 6! M^6}\right).$$

In the rest of this section, we estimate  $\mathcal{R}_3^+$ . By choosing  $L_1, L_2$  large enough (possibly depending on  $\varepsilon$ ) in (6.7) the contribution to  $\mathcal{R}_3^+$  from the first two error terms in (6.7) can be made as small as desired. We need to estimate the contribution from the last error term in (6.7). By the support of  $g$  we may assume  $x^2 = \frac{16\pi^2 n}{c^2} \ll N = T^{3+\varepsilon}$ . By our assumptions on  $M$  and  $T$  we then have

$$\frac{T|x|^3}{M^{18}} \ll \frac{T|x|}{M^9}.$$

Plugging in  $x = \frac{4\pi\sqrt{n}}{c}$  into  $\frac{T|x|}{M^9}$ , we estimate this error term contribution to  $\mathcal{R}_3^+$  in (6.4), using Weil's bound for the Kloosterman sum and the compact support of  $g$ . This error can be seen to be bounded by  $O\left(\frac{TN}{M^9}\right)$  which is smaller than  $O(T^{1+\varepsilon}M)$  by a power of  $T$  with our assumption  $T^{1/3+2\varepsilon} \leq M \leq \sqrt{T}$ . In the finite series (6.7) with our assumptions we also have

$$\frac{M^{2l-l_1} T^{4l_2-l_1}}{x^{l-l_1+3l_3}} \ll 1.$$

All the terms in (6.7) are similar, and can be estimated in a similar way, so we will only work with the first term. Following Li [19] we define

$$(6.8) \quad \begin{aligned} \tilde{\mathcal{R}}_3^+ &= \frac{i(i+1)MT}{\sqrt{2\pi}} \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{mn^{3/4}} g\left(\frac{m^2 n}{N}\right) \\ &\times \sum_{c \leq C_2/m} \frac{S(n, 1; c)}{\sqrt{c}} e\left(\frac{2\sqrt{n}}{c} - \frac{T^2 c}{4\pi^2 \sqrt{n}}\right) \widehat{k^*}\left(\frac{MTc}{2\pi^2 \sqrt{n}}\right). \end{aligned}$$

Li [19] points out here, that even with Weil's bound for  $S(n, 1; c)$  simple estimates for  $\tilde{\mathcal{R}}_3^+$  are too large. So we expand the Kloosterman sum  $S(n, 1; c)$  and use the Voronoi formula (Proposition 5.1) with

$$\psi(y) = y^{-\frac{3}{4}} g\left(\frac{m^2 n}{N}\right) e\left(\frac{2\sqrt{y}}{c} - \frac{T^2 c}{4\pi^2 \sqrt{y}}\right) \widehat{k^*}\left(\frac{MTc}{2\pi^2 \sqrt{y}}\right).$$

Following the discussion around Proposition 5.1 and Lemma 5.2 for  $c \leq C_2/m$  the assumptions of Lemma 5.2 are satisfied and so up to lower order terms, for  $x = n_2 n_1^2 / (c^3 m)$ ,

$$(6.9) \quad \Psi_0(x) = \pi^3 d_1 x^{2/3} \int_0^\infty e(u_1(y)) a(y) dy - \pi^3 d_1 x^{2/3} \int_0^\infty e(u_2(y)) a(y) dy$$

with

$$u_1(y) = \frac{2\sqrt{y}}{c} + 3(xy)^{1/3}, \quad u_2(y) = \frac{2\sqrt{y}}{c} - 3(xy)^{1/3}$$

and

$$(6.10) \quad a(y) = g\left(\frac{m^2 y}{N}\right) \widehat{k^*}\left(\frac{MTc}{2\pi^2 \sqrt{y}}\right) e\left(\frac{-T^2 c}{4\pi^2 \sqrt{y}}\right) y^{-13/12}.$$

Note that  $u_1$  has no stationary points; indeed simple calculus estimates give the first integral in (6.9) a negligible contribution to  $\tilde{\mathcal{R}}_3^+$ .

The second integral in (6.9) requires more analysis. If

$$x \geq 2 \frac{\sqrt{N}}{c^3 m} \quad \text{or} \quad x \leq \frac{2}{3} \frac{\sqrt{N}}{c^3 m},$$

then  $u_2'(y)$  will be effectively bounded away from zero, making the integral negligible by multiple integration by parts. Thus we assume the contrary in what follows, namely

$$(6.11) \quad \frac{2}{3} \frac{\sqrt{N}}{n_1^2} \leq n_2 \leq \frac{\sqrt{N}}{n_1^2}.$$

We have

$$(6.12) \quad \int_0^\infty e(u_2(y))a(y) dy = \int_{\frac{1}{4}x^2c^6}^{\frac{9}{2}x^2c^6} e(u_2(y))a(y) dy.$$

We explain the limits of integration. The compact support of the integral on the right side of equation (6.12) follows from the compact support of  $g$ , and so that of  $a$ . Further, recall  $x = n_2n_1^2/(c^3m)$ . As Li [19] points out, the stationary phase point of the integral in (6.12) is at  $y_0 = x^2c^6$ . The constants  $1/4$  and  $9/2$  in the limits of this integral give a segment that the support of  $a$  is contained in, since  $g \in C_c^\infty([1, 2])$ . In (6.10), from the support of  $g$ , and since  $\widehat{k^*}$  is a Schwartz function, we can assume

$$\frac{N}{m^2} \leq y \leq 2\frac{N}{m^2} \text{ and } \frac{MTc}{2\pi^2\sqrt{y}} \ll T^\varepsilon.$$

Using this information, simple calculus estimates give us

$$(6.13) \quad u_2^{(r)}(y) \ll T_1 M_1^{-r} \text{ for } r = 1, 2, \dots, 2n_0 + 3 \text{ and } a^{(r)}(y) \ll U_1 N_1^{-r} \text{ for } r = 0, 1, 2, \dots, 2n_0 + 1$$

for  $y$  in the segment.  $n_0 \in \mathbb{N}$  will be chosen in terms of  $\varepsilon_0$  later. Here

$$(6.14) \quad M_1 = \frac{N}{m^2}, \quad T_1 = \frac{\sqrt{N}}{cm}, \quad N_1 = \frac{N^{3/2}}{T^2 cm^3}, \quad U_1 = \left(\frac{N}{m^2}\right)^{-13/12}.$$

Further,  $u_2^{(2)}(y) \gg T_1 M_1^{-2}$  for  $y \in [\frac{1}{4}x^2c^6, \frac{9}{2}x^2c^6]$ . The condition  $N_1 \geq M_1/\sqrt{T_1}$  is then consistent with our assumption  $c \leq C_2/m$  when  $M \geq T^{1/3+2\varepsilon}$ .

Then, all assumptions are satisfied, and we apply Theorem 3.6 (where we take  $n = n_0$ ). The main term of the integral in (6.12) is

$$(6.15) \quad \frac{e(u_2(y_0) + 1/8)}{\sqrt{u_2''(y_0)}} \left( a(y_0) + \sum_{j=1}^{n_0} \varpi_{2j} \frac{(-1)^j (2j-1)!!}{(4\pi i \lambda_2)^j} \right),$$

where  $\varpi_{2j}$  are defined above and  $\lambda_2 = u_2''(y_0)/2$ . Notice we have used

$$\gamma - \alpha \asymp \beta - \gamma \asymp M_1,$$

with  $\alpha = \frac{1}{4}x^2c^6$ ,  $\beta = \frac{9}{2}x^2c^6$  and  $\gamma = y_0 = x^2c^6$ . To save time in estimates, notice there are no boundary terms here. This is due to the compact support of  $a$ , with itself and all of its derivatives zero at  $\frac{1}{4}x^2c^6$  and  $\frac{9}{2}x^2c^6$ . The sum of the four error terms in Theorem 3.6 can be simplified to

$$(6.16) \quad O\left(\frac{U_1 M_1^{2n_0+2}}{T_1^{n_0+1} N_1^{2n_0+1}}\right).$$

This estimate uses the current assumptions on  $c$  and  $m$ , and the size of  $N$  compared to  $T$ . Note that  $M_1 \gg N_1$ .

We need to estimate this error term, as well as error terms coming from the  $\varpi_{2j}$  terms which will be very similar. First we need a nifty estimate from Li [19]. Using the basic definitions, as Li points out (equation (4.22) of [19])

$$(6.17) \quad \sum_{0 \leq d \leq c}^* e\left(\frac{d}{c}\right) S(md, n_2; mc n_1^{-1}) = \sum_{u \pmod{mc n_1^{-1}}}^* S(0, 1 + un_1; c) e\left(\frac{n_2 \bar{u}}{mc n_1^{-1}}\right).$$

Here  $u\bar{u} \equiv 1 \pmod{mc n_1^{-1}}$  and

$$S(0, a; c) = \sum_{v \pmod{c}}^* e\left(\frac{av}{c}\right)$$

is the Ramanujan sum, which is  $\ll (a, c)$ . Then (6.17)

$$\ll \sum_{u \pmod{mcn_1^{-1}}}^* (1 + un_1, c) = \sum_{d|c} d \sum_{\substack{u \pmod{mcn_1^{-1}} \\ (1+un_1, c)=d}} 1 \ll \sum_{d|c} d \sum_{\substack{u \pmod{mcn_1^{-1}} \\ un_1 \equiv -1 \pmod{d}}} 1.$$

Now  $(n_1, d) = 1$  and so  $n_1^{-1}$  exists  $\pmod{d}$ , so the last inner sum above is over all  $u$  with  $0 \leq u < mc n_1^{-1}$  with  $u \equiv -n_1^{-1} \pmod{d}$ . The number of such terms is clearly

$$\asymp \frac{1}{d} \cdot \frac{mc}{n_1}.$$

Plugging this in above we see (6.17)

$$\ll \frac{mc}{n_1} \sum_{d|c} 1 \ll \frac{mc^{1+\varepsilon}}{n_1}.$$

We then have the important savings estimate

$$(6.18) \quad \sum_{u \pmod{mcn_1^{-1}}}^* S(0, 1 + un_1; c) e\left(\frac{n_2 \bar{u}}{mc n_1^{-1}}\right), \quad \sum_{u \pmod{mcn_1^{-1}}}^* S(0, 1 + un_1; c) \ll \frac{mc^{1+\varepsilon}}{n_1}$$

The following Lemma is specific to the weighted stationary phase integral (6.12).

**Lemma 6.2.** *Suppose we have an error term of the form  $O(c^\alpha T^\beta N^\gamma m^\delta)$  for specific numbers  $\alpha, \beta, \delta, \gamma$ , from the weighted stationary phase integral. Then the contribution of this error to  $\tilde{\mathcal{R}}_3^+$  is*

$$\ll M^{2/3-\delta-\varepsilon} T^{13/6+\beta+3\gamma+\delta/2+\varepsilon_1}$$

where  $\varepsilon_1$  is arbitrarily small, if  $\varepsilon$  is arbitrarily small. Here we assume  $\alpha + 2\varepsilon > -1/2$  and  $\delta - \alpha \geq 1/6$ .

*Proof.* Here  $\varepsilon_1$  will of course also depend on the numbers  $\alpha, \beta, \delta, \gamma$ . The contribution of this error to  $\tilde{\mathcal{R}}_3^+$  is (estimate (4.23) of Li [19])

$$\ll MT \sum_{m \leq C_2} m^{-2/3+\varepsilon} \sum_{c \leq C_2/m} c^{-1/2+\varepsilon} \sum_{n_1 | cm} n_1^{1/3} \sum_{n_2 \asymp \sqrt{N}/n_1^2} \frac{|A(n_1, n_2)|}{n_2^{1/3}} (c^\alpha T^\beta N^\gamma m^\delta).$$

Note that the innermost sum is over

$$\frac{2\sqrt{N}}{3n_1^2} \leq n_2 \leq 2\frac{\sqrt{N}}{n_1^2}$$

(see (6.11)). Also note Li seems [19] to have used the estimate  $(mc)^{1+\varepsilon}$  instead of the estimate  $mc^{1+\varepsilon}/n_1$  from the estimate (6.18). Since the sum on  $n_1$  is a divisor sum, this is not an issue here. Using the estimates for  $|A(n_1, n_2)|$  (see (5.3)), and partial summation one has

$$\sum_{n_2 \asymp \sqrt{N}/n_1^2} \frac{|A(n_1, n_2)|}{n_2^{1/3}} \ll n_1 \left(\frac{\sqrt{N}}{n_1^2}\right)^{2/3}.$$

Since the number of divisors of  $cm$  is  $\ll (cm)^\varepsilon$  this simplifies the contribution to

$$\ll MT^{1+\beta} N^{1/3+\gamma} \sum_{m \leq C_2} m^{-2/3+2\varepsilon+\delta} \sum_{c \leq C_2/m} c^{-1/2+2\varepsilon+\alpha}.$$

From a calculus estimate, we have

$$\sum_{c \leq C_2/m} c^{-1/2+2\varepsilon+\alpha} \ll \left(\frac{C_2}{m}\right)^{1/2+2\varepsilon+\alpha}.$$

Plugging this in, and using  $C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon}M}$  we have

$$MT^{1+\beta} N^{\frac{1}{3}+\gamma} \left(\frac{\sqrt{N}}{T^{1-\varepsilon}M}\right)^{\frac{1}{2}+2\varepsilon+\alpha} \sum_{m \leq C_2} m^{-\frac{7}{6}+\delta-\alpha}.$$

Now, if  $\delta - \alpha > \frac{1}{6}$  we have

$$\sum_{m \leq C_2} m^{-\frac{7}{6} + \delta - \gamma} \ll C_2^{-\frac{1}{6} + \delta - \alpha}.$$

If  $\delta - \alpha = \frac{1}{6}$ , since  $C_2 \leq T^{\frac{1}{6}}$ , with our assumption on  $M$  and  $T$ , we take the sum on  $m$  to be  $\ll M^\varepsilon$ .

Combining both cases,  $\delta - \alpha > \frac{1}{6}$  and  $\delta - \alpha = \frac{1}{6}$ , we can take the bound

$$M^{\frac{2}{3} - \delta - 2\varepsilon} T^{1 + \beta - (1 - \varepsilon)(\frac{1}{3} + \delta + 2\varepsilon)} N^{\frac{1}{2} + \gamma + \frac{1}{2}\delta + \varepsilon}.$$

Now plugging in  $N = T^{3+\varepsilon}$  gives our Lemma.

For use below, we can take

$$\varepsilon_1 = \varepsilon \left( \frac{23}{6} + \frac{3}{2}\delta + \gamma \right) + 3\varepsilon^2.$$

□

The error term (6.16) can be seen to be

$$O \left( c^{3n_0+2} T^{4n_0+2} N^{-\frac{3}{2}n_0 - \frac{13}{12}} m^{3n_0 + \frac{13}{6}} \right).$$

By Lemma 6.2 the contribution to  $\tilde{\mathcal{R}}_3^+$  of this term is  $\ll T^{1+\varepsilon} M$  if

$$(6.19) \quad M \geq T^{\frac{n_0+1+\varepsilon_1-\varepsilon}{3n_0+\frac{1}{2}+\varepsilon}}.$$

$$(6.20) \quad \varepsilon_1 = \varepsilon(6 + 3n_0) + 3\varepsilon^2$$

clearly depends on  $n_0$ , which we will choose later depending on  $\varepsilon_0$ . Notice that if  $n_0 = 1/2$ , we pick up the  $\frac{3}{8}$  constant of LI [19].

We now need to deal with the  $\varpi_{2j}$  terms in (6.15). Recall the expression for  $\varpi_{2j}$  in equation (3.38). Here we take  $2 \leq 2j \leq 2n_0$ . One can see from (3.38) that the main term from  $\varpi_{2j}$  is  $a^{(2j)}(y_0)$ . ( $a$  given in equation (6.10), and  $u_2$ , above (6.10), take the place of  $g$  and  $f$  in Theorem 3.6. Further  $y_0$  takes the place of  $\gamma$ .) Using the estimates in (6.13) along with  $u_2''(y_0) \gg T_1/M_1^2$ , and along with our current assumptions on  $c$  and  $m$  we have

$$(6.21) \quad \varpi_{2j} - a^{(2j)}(y_0) = O \left( \frac{U_1}{M_1 N_1^{2j-1}} \right)$$

The constant ultimately depends on  $n_0$  and we have used  $M_1 \gg N_1$ . To estimate this error term contribution to  $\tilde{\mathcal{R}}_3^+$ , we must divide by  $\lambda_2^{j+\frac{1}{2}}$  and sum over  $j$ . (See (6.15).) Since  $y_0 \asymp \frac{N}{m^2}$ , we have  $\lambda_2 \asymp \frac{m^3}{cN^{\frac{3}{2}}}$ . We have then that this contribution is

$$\ll \left( \frac{N}{m^2} \right)^{-\frac{25}{12}} \left( \frac{T^2 c m^3}{N^{\frac{3}{4}}} \right)^{2j-1} \left( \frac{c N^{\frac{3}{2}}}{m^3} \right)^{j+\frac{1}{2}} = O \left( c^{3j-\frac{1}{2}} T^{4j-2} N^{-\frac{3}{2}j+\frac{1}{6}} m^{3j-\frac{1}{3}} \right).$$

By Lemma 6.2 this term is

$$(6.22) \quad \ll T^{1+\varepsilon} M \text{ if } M \geq T^{\frac{j-\frac{1}{2}+\varepsilon_1-\varepsilon}{3j+\varepsilon}}.$$

Here  $\varepsilon_1 = \varepsilon(3j + \frac{7}{2}) + 3\varepsilon^2$ . So we have

$$\frac{j - \frac{1}{2} + \varepsilon_1 - \varepsilon}{3j + \varepsilon} \leq \frac{1}{3} - \frac{1}{6j} + 3\varepsilon$$

for  $j \geq 1$ . Thus the condition on  $M$  in (6.22) is always true for  $M \geq T^{\frac{1}{3}+3\varepsilon}$ .

We must now estimate the  $a^{(2j)}(y_0)$  term in  $\varpi_{2j}$  in (6.15). For convenience, we recall that

$$a(y) = g \left( \frac{m^2 y}{N} \right) \widehat{k^*} \left( \frac{M T c}{2\pi^2 \sqrt{y}} \right) e \left( \frac{-T^2 c}{4\pi^2 \sqrt{y}} \right) y^{-13/12}.$$

Let  $i_1$  be the number of times  $g\left(\frac{m^2 y}{N}\right)$  is differentiated plus the number of times  $y^{-13/12}$  is differentiated. So at every differentiation either the factor  $\frac{m^2}{N}$  comes out, or up to a constant, the factor  $\frac{1}{y}$  comes out. Notice that  $\frac{1}{y} \asymp \frac{m^2}{N}$ . Let  $i_2$  be the number of times  $\widehat{k^*}\left(\frac{MTc}{2\pi^2\sqrt{y}}\right)$  is differentiated, and put  $i_3$  to be the number of times  $e\left(\frac{-T^2 c}{4\pi^2\sqrt{y}}\right)$  is differentiated. Then  $i_1 + i_2 + i_3 = 2j$ , and neglecting coefficients (which ultimately depend on  $n_0$ ),  $a^{(2j)}(y_0)$  is the sum over all combinatorial possibilities of

$$\left(\frac{N}{m^2}\right)^{-\frac{13}{12}-i_1} \left(\frac{MTcm^3}{N^{\frac{3}{2}}}\right)^{i_2} \left(\frac{T^2 cm^3}{N^{\frac{3}{2}}}\right)^{i_3}.$$

The main term is when  $i_3 = 2j$  and we will estimate this separately, below. So we can assume in all terms, now, that  $i_1 + i_2 \geq 1$ . To estimate this error term, which is all but one term in  $a^{(2j)}(y_0)$ , as before, in (6.15), we must divide by  $\lambda_2^{j+\frac{1}{2}}$  where  $\lambda_2 \asymp \frac{m^3}{cN^{\frac{3}{2}}}$  with our assumption on  $y_0$ . We have then a sum of error terms which are all

$$O\left(M^{i_2} c^{j+i_2+i_3+\frac{1}{2}} T^{i_2+i_3} N^{\frac{3}{2}j-i_1-\frac{3}{2}i_2-\frac{3}{2}i_3-\frac{1}{3}} m^{-3j+2i_1+3i_2+3i_3+\frac{2}{3}}\right).$$

Using  $i_3 = 2j - i_1 - i_2$ , by Lemma 6.2 this error term can be seen to be

$$(6.23) \quad \ll M^{-3j+i_1+i_2-\varepsilon} T^{j-i_1-i_2+\frac{3}{2}+\varepsilon} \leq T^{1+\varepsilon} M \text{ if } M \geq T^{\frac{j-i_1-i_2+\frac{1}{2}+\varepsilon_1-\varepsilon}{3j-i_1-i_2+1+\varepsilon}}.$$

Here

$$\varepsilon_1 = \varepsilon(3j - i_1 + \frac{9}{2}) + 3\varepsilon^2.$$

Now

$$\frac{j - i_1 - i_2 + \frac{1}{2} + \varepsilon_1 - \varepsilon}{3j - i_1 - i_2 + 1 + \varepsilon} \leq \frac{j - i_1 - i_2 + \frac{1}{2}}{3j - i_1 - i_2 + 1} + 10\varepsilon$$

We are assuming  $1 \leq i_1 + i_2 \leq 2j$  with  $j \geq 1$ , and so

$$\frac{j - i_1 - i_2 + \frac{1}{2}}{3j - i_1 - i_2 + 1} + 10\varepsilon \leq \frac{1}{3} - \frac{1}{6j} + 10\varepsilon.$$

Consequently, the latter condition on  $M$  in (6.23) is always true for  $M \geq T^{\frac{1}{3}+10\varepsilon}$ .

This leaves the main term of  $a^{(2j)}(y_0)$  (where  $i_3 = 2j$  and  $i_1 = i_2 = 0$ ) which is

$$(6.24) \quad \alpha_j \left(\frac{T^2 c}{y^{\frac{3}{2}}}\right)^{2j} g\left(\frac{m^2 y}{N}\right) \widehat{k^*}\left(\frac{MTc}{2\pi^2\sqrt{y}}\right) e\left(\frac{-T^2 c}{4\pi^2\sqrt{y}}\right) y^{-13/12} \Big|_{y_0} =: a_{2j}(y_0).$$

Here, the constant  $\alpha_j$  depends on  $j$  which ultimately can be bounded in terms of  $n_0$ . If we estimate this similarly, we will get an estimate similar to (6.19) with  $2j$  replacing  $n_0$ . Instead, we will apply the Voronoi formula to (6.24). This is very similar to Li [19], in applying the Voronoi formula a second time, but only to the main term  $\frac{e(u_2(y_0)+1/8)}{\sqrt{u_2''(y_0)}} a(y_0)$  in (6.15). It appears that the term  $\left(\frac{T^2 c}{y^{\frac{3}{2}}}\right)^{2j}$  in (6.24) for  $1 \leq j \leq n_0$  is on average  $\asymp 1$  in summing over  $m$  and  $c$ , and so we do not improve upon the second application of Voronoi to the term for just  $j = 0$ .

Recall that in (6.9) we have  $x = n_2 n_1^2 / (c^3 m)$  and  $y_0 = x^2 c^6 = \frac{n_2^2 n_1^4}{m^2}$ . Further,  $\lambda_2 = \frac{1}{12} c^{-1} y_0^{-\frac{3}{2}}$ . The contribution to  $\widetilde{\mathcal{R}}_3^+$  of  $a_{2j}(y_0)$  in (6.8) is then  $\asymp \widetilde{\mathcal{R}}_{3,j}^+$  where

$$(6.25) \quad \begin{aligned} \widetilde{\mathcal{R}}_{3,j}^+ &= MT \sum_{m \leq C_2} \frac{1}{m} \sum_{c \leq C_2/m} \frac{1}{c^{\frac{3}{2}}} \sum_{n_1 | cm} \sum_{n_2 > 0} c \frac{A(n_1, n_2)}{n_1 n_2} \sum_{u \pmod{mc n_1^{-1}}}^* S(0, 1 + un_1; c) \\ &\quad \times e\left(\frac{n_2 \bar{u}}{mc n_1^{-1}}\right) e(-xc^2) x^{\frac{2}{3}} \frac{a_{2j}(y_0)}{\lambda_2^{j+\frac{1}{2}}}. \end{aligned}$$

Inserting what  $x$ ,  $y_0$ , and  $\lambda_2$  are in terms of  $n_1$ ,  $n_2$ ,  $c$  and  $m$  we have

$$(6.26) \quad \tilde{\mathcal{R}}_{3,j}^+ = MT^{4j+1} \sum_{m \leq C_2} m^{3j-1} \sum_{c \leq C_2/m} c^{3j-1} \sum_{n_1 | cm} \frac{1}{n_1^{6j+1}} \sum_{n_2 > 0} \frac{A(n_2, n_1)}{n_2^{3j+1}} \sum_{u \pmod{mcn_1^{-1}}}^* S(0, 1 + un_1; c) \\ \times e\left(\frac{n_2 \bar{u}}{mcn_1^{-1}}\right) e\left(-\frac{n_2 n_1^2}{cm}\right) g\left(\frac{n_2^2 n_1^4}{N}\right) \widehat{k^*}\left(\frac{MTcm}{2\pi^2 n_2 n_1^2}\right) e\left(-\frac{T^2 cm}{4\pi^2 n_2 n_1^2}\right).$$

We can switch the sums over  $n_2$  and  $u$ , pull out  $S(0, 1 + un_1; c)$  which does not depend on  $n_2$  and then the inner sum on  $n_2$  is

$$(6.27) \quad \sum_{n_2 > 0} A(n_2, n_1) e\left(\frac{n_2 u'}{c'}\right) b_j(n_2)$$

where

$$b_j(y) = \frac{1}{y^{3j+1}} g\left(\frac{y^2 n_1^4}{N}\right) \widehat{k^*}\left(\frac{MTcm}{2\pi^2 y n_1^2}\right) e\left(-\frac{T^2 cm}{4\pi^2 y n_1^2}\right)$$

and

$$\frac{u'}{c'} = \frac{\bar{u} - n_1}{mcn_1^{-1}}, \text{ with } (u'c') = 1 \text{ and } c' | mcn_1^{-1}$$

We now apply the Voronoi formula for  $\text{GL}(3)$  a second time. (See (4.25) of Li [19]. Indeed Li [19] had the insight to use Lapid's Theorem as well as applying Voronoi twice.) We have:

$$(6.28) \quad \sum_{n_2 \geq 1} A(n_1, n_2) e\left(\frac{n_2 u'}{c'}\right) b(n_2) \\ = \frac{c'}{4\pi^{5/2} i} \sum_{l_1 | c'n_1} \sum_{l_2 > 0} \frac{A(l_2, l_1)}{l_1 l_2} S(n_1 \bar{u}', l_2; n_1 c' l_1^{-1}) B_{0,1}^0\left(\frac{l_1^2 l_2}{c'^3 n_1}\right) \\ + \frac{c'}{4\pi^{5/2} i} \sum_{l_1 | c'n_1} \sum_{l_2 > 0} \frac{A(l_2, l_1)}{l_1 l_2} S(n_1 \bar{u}', -l_2; n_1 c' l_1^{-1}) B_{0,1}^1\left(\frac{l_1^2 l_2}{c'^3 n_1}\right).$$

(We followed Li [19] in using the notation  $B$  rather than  $\Psi$ .) Put  $x = l_2 l_1^2 / (c'^3 n_1)$ . We apply Lemma 5.2 to  $B_0(x)$  (the growth assumption is satisfied) which is, up to a negligible amount and lower order terms (up to a constant)

$$(6.29) \quad x^{2/3} \int_0^\infty e(v_2(y)) q_j(y) dy$$

where

$$v_2(y) = -3(xy)^{1/3} - \frac{T^2 cm}{4\pi^2 y n_1^2}$$

and

$$q_j(y) = y^{-3j-\frac{4}{3}} g\left(\frac{y^2 n_1^4}{N}\right) \widehat{k^*}\left(\frac{MTcm}{2\pi^2 y n_1^2}\right).$$

See equation (4.26) of Li [19]. We need only consider the case

$$\frac{T^6 c^3 m^3 n_1^2}{10^3 \pi^6 N^2} \leq x \leq \frac{T^6 c^3 m^3 n_1^2}{10 \pi^6 N^2}.$$

Thus

$$(6.30) \quad x = \frac{l_2 l_1^2}{c'^3 n_1} \asymp \frac{T^6 c^3 m^3 n_1^2}{\pi^6 N^2}.$$

By the compact support of  $g$ , we may assume the integral (6.29) is taken over a compact segment in  $y$  so that

$$1 \leq \frac{y^2 n_1^4}{N} \leq 2.$$

With these assumptions, we have

$$|v_2''(y)| \gg \frac{T^2 c m n_1^4}{N^{3/2}}.$$

The variation of  $q_j$  over this interval can be seen to be  $\ll y_0^{-3j-\frac{4}{3}} T^\varepsilon$ . This computation uses basic estimates with simple calculus. Also needed, is that

$$y \asymp \frac{\sqrt{N}}{n_1^2}, \quad n_1 \leq cm \leq C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon} M}, \quad \text{and} \quad M \geq T^{1/3+2\varepsilon}.$$

Then, by the second derivative test (see Huxley [10]), we have by (6.30) that

$$(6.31) \quad B_0(x) \ll \left( \frac{l_2 l_1^2}{c'^3 n_1} \right)^{\frac{2}{3}} \left( \frac{T^2 c m n_1^4}{N^{3/2}} \right)^{-1/2} \left( \frac{\sqrt{N}}{n_1^2} \right)^{-3j-\frac{4}{3}} T^\varepsilon \ll T^{3+\varepsilon} c^{3/2} N^{-\frac{3}{2}j-\frac{5}{4}} n_1^{6j+2} m^{3/2}.$$

Put

$$L_2 = \frac{T^6 c^3 m^3 n_1^2 c'^3}{\pi^6 N^2 l_1^2}.$$

Combining (6.31), (6.26), and (6.28) we see

$$\begin{aligned} \tilde{\mathcal{R}}_{3,j}^+ &\ll MT^{4j+1} \sum_{m \leq C_2} m^{3j-1} \sum_{c \leq C_2/m} c^{3j-1} \sum_{n_1 | cm} \frac{1}{n_1^{6j+1}} \sum_{u \pmod{mc n_1^{-1}}} (1 + un_1, c) c' \\ &\times \sum_{l_1 | c' n_1} \sum_{l_2 \asymp L_2} \frac{|A(l_1, l_2)|}{l_1 l_2} \times \left( \frac{n_1 c'}{l_1} \right) (T^{3+\varepsilon} c^{\frac{3}{2}} N^{-\frac{3}{2}j-\frac{5}{4}} n_1^{6j+2} m^{\frac{3}{2}}). \end{aligned}$$

Here  $l_2 \asymp L_2$  means  $L_2/10^3 \leq l_2 \leq L_2/10$ . Also, we have used the trivial bound for the Kloosterman sum:

$$(6.32) \quad \left| S\left(n_1 \bar{u}, l_2; \frac{n_1 c'}{l_1}\right) \right| \leq \frac{n_1 c'}{l_1}.$$

Using the estimate (6.18) and that  $c' \leq mc/n_1$ , we see

$$\tilde{\mathcal{R}}_{3,j}^+ \ll N^{-\frac{3}{2}j-\frac{5}{4}} MT^{4+\varepsilon} \sum_{m \leq C_2} m^{3j+\frac{7}{2}} \sum_{c \leq C_2/m} c^{3j+\frac{7}{2}+\varepsilon} \sum_{n_1 | cm} \frac{1}{n_1} \sum_{l_1 | c' n_1} \frac{1}{l_1^2} \sum_{l_2 \asymp L_2} \frac{|A(l_1, l_2)|}{l_2}.$$

Now

$$\sum_{l_2 \asymp L_2} \frac{|A(l_1, l_2)|}{l_2} \ll l_1 L_2^\varepsilon \ll l_1^{1-2\varepsilon} \frac{T^{6\varepsilon} c^{6\varepsilon} m^{6\varepsilon}}{n_1^\varepsilon N^{2\varepsilon}}.$$

Further

$$\sum_{l_1 | c' n_1} \frac{1}{l_1^{1+2\varepsilon}} = O(\varepsilon^{-1})$$

by comparison to the Riemann zeta function evaluated at  $1 + 2\varepsilon$ . Similarly

$$\sum_{n_1 | cm} \frac{1}{n_1^{1+\varepsilon}} = O(\varepsilon^{-1}).$$

Consequently,

$$\tilde{\mathcal{R}}_{3,j}^+ \ll N^{-\frac{3}{2}j-\frac{5}{4}} MT^{4j+4+7\varepsilon} \sum_{m \leq C_2} m^{3j+\frac{7}{2}+6\varepsilon} \sum_{c \leq C_2/m} c^{3j+\frac{7}{2}+7\varepsilon}.$$

Simple calculus and similar estimates then give us

$$\tilde{\mathcal{R}}_{3,j}^+ \ll N^{-\frac{3}{2}j-\frac{5}{4}} MT^{4j+4+7\varepsilon} C_2^{3j+\frac{9}{2}+7\varepsilon}.$$

Plugging in

$$N = T^{3+\varepsilon} \quad \text{and} \quad C_2 = \frac{\sqrt{N}}{T^{1-\varepsilon} M},$$

we see this last estimate for  $\tilde{\mathcal{R}}_{3,j}^+$  is

$$(6.33) \quad \ll M^{-3j-\frac{7}{2}-7\varepsilon} T^{j+\frac{5}{2}+\varepsilon_2}.$$



Here

$$\varepsilon_2 = \varepsilon(3j + 16) + \frac{21}{2}\varepsilon^2$$

This final term (6.33) is  $\leq MT^{1+\varepsilon}$  if

$$(6.34) \quad M \geq T^{\frac{j + \frac{3}{2} + \varepsilon_2 - \varepsilon}{3j + \frac{9}{2} + 7\varepsilon}}.$$

Now  $0 \leq j \leq n_0$ , and

$$\frac{j + \frac{3}{2} + \varepsilon_2 - \varepsilon}{3j + \frac{9}{2} + 7\varepsilon} \leq \frac{1}{3} + \frac{3j}{3j + \frac{9}{2}}\varepsilon + \frac{15}{3j + \frac{9}{2}}\varepsilon + \frac{21}{6j + 9}\varepsilon^2 \leq \frac{1}{3} + 7\varepsilon.$$

Thus (6.34) is always true for  $M \geq T^{\frac{1}{3} + 7\varepsilon}$ .

Ultimately, given  $0 < \varepsilon_0 \leq 1$ , we need to choose  $n_0$  so that the error term (6.19) is smaller than  $O(T^{1+\varepsilon}M)$ .

We recall: we need

$$(6.35) \quad M \geq T^{\frac{n_0 + 1 + \varepsilon_1 - \varepsilon}{3n_0 + \frac{5}{2} + \varepsilon}}, \text{ with } \varepsilon_1 = \varepsilon(6 + 3n_0) + 3\varepsilon^2.$$

Recall,  $n_0 \geq 2$  and with  $\varepsilon \leq 1$  we have

$$\frac{\varepsilon_1 - \varepsilon}{3n_0 + \frac{5}{2} + \varepsilon} \leq 2\varepsilon \leq \frac{\varepsilon_0}{2}.$$

We select  $n_0$  so that

$$\frac{n_0 + 1}{3n_0 + \frac{5}{2}} \leq \frac{1}{3} + \frac{\varepsilon_0}{2}.$$

Consequently, for this  $n_0$ , the condition on  $M$  in (6.35) is satisfied for  $M \geq T^{\frac{1}{3} + \varepsilon_0}$ . Further the constants we ignored in this section depend only on  $n_0$ , and thus only on  $\varepsilon_0$ .

## 7. $K$ -BESSEL FUNCTION TERMS

Following Li [19] we split  $\mathcal{R}^-$  into  $\mathcal{R}_1^- + \mathcal{R}_2^-$  with

$$(7.1) \quad \mathcal{R}_1^- = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c \geq C/m} c^{-1} S(n, -1; c) H_{m, n}^-\left(\frac{4\pi\sqrt{n}}{c}\right),$$

$$(7.2) \quad \mathcal{R}_2^- = \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c \leq C/m} c^{-1} S(n, -1; c) H_{m, n}^-\left(\frac{4\pi\sqrt{n}}{c}\right),$$

where we take

$$(7.3) \quad C = \sqrt{N} + T.$$

In estimating  $\mathcal{R}_1^-$ , one can express the  $K$ -Bessel function in terms of the  $I$ -Bessel function. Set  $\sigma = 100$ . Then the estimates for the  $I$ -Bessel function, along with Li's previous estimates of  $V$  (see (4.7) and (5.6) of Li [19]) give (using the trivial bound for the Kloosterman sum)

$$\mathcal{R}_1^- \ll MT^{\sigma+1+\varepsilon} \sum_{m \leq \sqrt{2N}} \frac{1}{m^{1+2\sigma}} \sum_{n \leq \frac{2N}{m^2}} \frac{A(m, n)}{n^{\frac{1}{2}}} \sum_{c \geq C/m} \frac{1}{c^{2\sigma}} e^{4\pi\frac{\sqrt{n}}{c}}.$$

Using  $n \leq \frac{2N}{m^2}$  and  $c \geq C/m$  we see that  $e^{4\pi\frac{\sqrt{n}}{c}} \ll 1$ . Further,

$$\sum_{c \geq C/m} \frac{1}{c^{2\sigma}} \ll \left(\frac{C}{m}\right)^{1-2\sigma} \text{ and } \sum_{n \leq \frac{2N}{m^2}} \frac{A(m, n)}{n^{\frac{1}{2}}} \ll m \left(\frac{2N}{m^2}\right)^{\frac{1}{2}}.$$

Plugging this in above, and noting the sum over  $m$  converges, we have

$$\mathcal{R}_1^- \ll \sqrt{N} M T^{\sigma+1+\varepsilon} C^{1-2\sigma} \ll 1$$

for  $\varepsilon$  sufficiently small. Notice this bound holds for  $T^\varepsilon \leq M \leq T^{1-\varepsilon}$ .

Following the derivation in Li [19], up to a negligible term, we can write

$$H_{m,n}^-(x) = H_{m,n}^{-,1}(x) + H_{m,n}^{-,2}(x)$$

where

$$\begin{aligned} H_{m,n}^{-,j}(x) &= \frac{4M^j T^{2-j}}{\pi} \int_{\mathbb{R}} \int_{|\zeta| \leq T^\varepsilon} t^{j-1} e^{-t^2} V(m^2 n, tM + T) \\ &\quad \times \cos(x \sinh \zeta) e\left(-\frac{(tM + T)\zeta}{\pi}\right) dt d\zeta, \end{aligned}$$

for  $j = 1, 2$ .  $H_{m,n}^{-,2}(x)$  is a lower order term. We only work with  $H_{m,n}^{-,1}(x)$ , since the analysis with  $H_{m,n}^{-,2}(x)$  is similar. Up to a negligible amount, we can write

$$H_{m,n}^{-,1}(x) = 4Y_{m,n}(x),$$

where

$$Y_{m,n}(x) = \frac{Y_{m,n}^*(x) + Y_{m,n}^*(-x)}{2},$$

with

$$(7.4) \quad Y_{m,n}^*(x) = T \int_{\mathbb{R}} \widehat{k^*}(\zeta) e\left(-\frac{T\zeta}{M} + \frac{x}{2\pi} \sinh \frac{\zeta\pi}{M}\right) d\zeta.$$

The part of the integral over  $|\zeta| \geq M^{\varepsilon/2}$  is negligible. Further, with this assumption, it can be shown by integration by parts, that  $Y_{m,n}^*(x)$  is negligible unless

$$(7.5) \quad \frac{T}{100} \leq |x| \leq 100T \text{ and } \frac{x}{M^3} \ll T^{-\varepsilon},$$

which we now assume. Recall  $M \geq T^{\frac{1}{3}+2\varepsilon}$ . Thus, the sum over  $c$  in (7.2) for which  $c \geq \frac{400\pi\sqrt{N}}{Tm}$  or  $c \leq \frac{\sqrt{2}\pi\sqrt{N}}{25Tm}$  is negligible. We thus may assume  $\frac{\sqrt{2}\pi\sqrt{N}}{25Tm} \leq c \leq \frac{400\pi\sqrt{N}}{Tm}$  and we will denote this by  $c \asymp \frac{\sqrt{N}}{Tm}$ .

Using one more nonzero term in the Taylor expansion than Li [19], estimating, we have

$$\begin{aligned} Y_{m,n}^*(x) &= T \int_{\mathbb{R}} \widehat{k^*}(\zeta) e\left(-\frac{T\zeta}{M} + \frac{x\zeta}{2M} + \frac{\pi^2 x \zeta^3}{12M^3} + \frac{\pi^4 x \zeta^5}{240M^5} + \frac{\pi^6 x \zeta^7}{2 \cdot 7!M^7}\right) d\zeta \\ &\quad + O\left(T \int_{\mathbb{R}} |\widehat{k^*}(\zeta)| \frac{|\zeta|^9 |x|}{M^9}\right). \end{aligned}$$

Now, expanding

$$e\left(\frac{\pi^2 x \zeta^3}{12M^3} + \frac{\pi^4 x \zeta^5}{240M^5} + \frac{\pi^6 x \zeta^7}{2 \cdot 7!M^7}\right)$$

into a Taylor series of order  $L_2$  (which could depend on  $\varepsilon$ ) we have

$$\begin{aligned} Y_{m,n}^*(x) &= T \int_{\mathbb{R}} \widehat{k^*}(\zeta) e\left(-\frac{(2T-x)\zeta}{2M}\right) \\ &\quad \times \sum_{j_1+j_2+j_3 \leq L_2} d_{j_1,j_2,j_3} \left(\frac{x\zeta^3}{M^3}\right)^{j_1} \left(\frac{x\zeta^5}{M^5}\right)^{j_2} \left(\frac{x\zeta^7}{M^7}\right)^{j_3} d\zeta \\ &\quad + O\left(\frac{T|x|^{L_2+1}}{M^{3L_2+3}} + \frac{T|x|}{M^9}\right), \end{aligned}$$

where  $d_{j_1,j_2,j_3}$  are constants with  $d_{0,0,0} = 1$  with the sum taken over  $j_1 \geq 0$ ,  $j_2 \geq 0$ , and  $j_3 \geq 0$ . It follows that

$$(7.6) \quad Y_{m,n}^*(x) = T \sum_{j_1+j_2+j_3 \leq L_2} \frac{d_{j_1,j_2,j_3} \cdot x^{j_1+j_2+j_3}}{(2\pi i M)^{3j_1+5j_2+7j_3}} k^{*(3j_1+5j_2+7j_3)} \left(\frac{x-2T}{2M}\right) + O\left(\frac{T|x|^{L_2+1}}{M^{3L_2+3}} + \frac{T|x|}{M^9}\right).$$

We take  $L_2$  large enough (possibly depending on  $\varepsilon$ ) so that the error term coming from the

$$O\left(\frac{T|x|^{L_2+1}}{M^{3L_2+3}}\right)$$

above is negligible, or rather has as fast inverse polynomial decay as desired. (Recall  $\frac{x}{M^3} \ll T^{-\varepsilon}$ .) The contribution to  $\mathcal{R}_2^-$  coming from the error term  $O(T|x|/M^9)$  can be seen to be bounded by

$$\frac{T^2}{M^9} \sum_{m \leq \sqrt{2N}} \frac{1}{m} \sum_{n \leq 2N/m^2} \frac{|A(m, n)|}{n^{\frac{1}{2}}} \sum_{c \leq C/m} \frac{|S(n, -1; c)|}{c}.$$

Using Weil's bound for  $S(n, -1; c)$  we see

$$\sum_{c \leq C/m} \frac{|S(n, -1; c)|}{c} \ll \left(\frac{C}{m}\right)^{\frac{1}{2}+\varepsilon}.$$

Estimating similarly to the above, we see this contribution is

$$\ll \frac{T^2}{M^9} C^{\frac{1}{2}+\varepsilon} \sqrt{N} = \frac{T^{2+\frac{3}{2}+\frac{3}{4}+\varepsilon}}{M^9}.$$

The above is  $\ll T^{1+\varepsilon}M$  by a power of  $T$  for  $M \geq T^{\frac{1}{3}+2\varepsilon}$ .

We take the leading term in the finite series for  $Y_{m,n}^*(x)$  in (7.6), as the terms with higher derivatives of  $k^*$  can be handled in the same way. It follows we need to bound

$$(7.7) \quad \tilde{\mathcal{R}}_2^- = T \sum_{m \geq 1} \sum_{n \geq 1} \frac{A(m, n)}{(m^2 n)^{\frac{1}{2}}} g\left(\frac{m^2 n}{N}\right) \sum_{c \asymp \frac{\sqrt{N}}{Tm}} \frac{S(n, -1; c)}{c} k^*\left(\frac{4\pi\sqrt{n}/c - 2T}{2M}\right).$$

We sum over  $n$  using the Voronoi formula on  $GL(3)$ . Here the smooth function of compact support is

$$r(y) = g\left(\frac{m^2 y}{N}\right) k^*\left(\frac{4\pi\sqrt{y}/c - 2T}{2M}\right) y^{-\frac{1}{2}}.$$

As in Li [19] we only consider  $R_0(x)$  (see (5.11) of [19]), which for  $x = n_2 n_1^2 / (c^3 m)$ , is (up to lower order terms)

$$R_0(x) = 2\pi^4 x i \int_0^\infty r(y) \frac{d_1 \sin(6\pi(xy)^{\frac{1}{3}})}{\pi(xy)^{\frac{1}{3}}} dy,$$

since

$$x \frac{N}{m^2} \gg \frac{N}{C^3} = T^{\frac{3}{2}-\varepsilon}.$$

Li [19] states that (in an equivalent form) if

$$n_2 \gg \frac{N^{\frac{1}{2}} T^\varepsilon}{M^3 n_1^2},$$

then

$$\frac{r'(y)}{x^{\frac{1}{3}} y^{-\frac{2}{3}}} \ll T^{-\varepsilon}.$$

For this assumption on  $n_2$ , the integral term in  $R_0$  as well as the contribution to  $\tilde{\mathcal{R}}_2^-$  is found to be negligible.

Thus, we may assume

$$n_2 \ll \frac{N^{\frac{1}{2}} T^\varepsilon}{M^3 n_1^2}.$$

Now,  $r(y)$  is negligible unless

$$\left| \frac{2\pi\sqrt{y}/c - T}{M} \right| \leq T^\varepsilon.$$

This gives us an interval of width  $\ll T^{1+\varepsilon} M c^2$  where  $y \asymp N/m^2$ , and so

$$R_0(x) \ll \left(\frac{n_2 n_1^2}{c^3 m}\right)^{\frac{2}{3}} \left(\frac{N}{m^2}\right)^{-\frac{5}{6}} T^{1+\varepsilon} M c^2.$$

Using this estimate along with (6.18) it follows that

$$\begin{aligned} \tilde{\mathcal{R}}_2^- &\ll T \sum_{m \leq \sqrt{2N}} \sum_{c \asymp \frac{\sqrt{N}}{Tm}} \sum_{n_1 | cm} \sum_{n_2 \ll \sqrt{NT^\varepsilon}/(M^3 n_1^2)} \frac{|A(n_1, n_2)|}{n_1 n_2} \frac{mc^{1+\varepsilon}}{n_1} \\ &\quad \times \left( \frac{n_2 n_1^2}{c^3 m} \right)^{\frac{2}{3}} \left( \frac{N}{m^2} \right)^{-\frac{5}{6}} T^{1+\varepsilon} M c^2 \\ &= T^{2+\varepsilon} M N^{-\frac{5}{6}} \sum_{m \leq \sqrt{2N}} m \sum_{c \asymp C/m} c^{1+\varepsilon} \sum_{n_1 | cm} \frac{1}{n_1^{\frac{2}{3}}} \sum_{n_2 \ll \sqrt{NT^\varepsilon}/(M^3 n_1^2)} \frac{|A(n_1, n_2)|}{n_2^{\frac{1}{3}}}. \end{aligned}$$

Estimating similarly to the last section, the inner sum

$$\sum_{n_2 \ll \sqrt{NT^\varepsilon}/(M^3 n_1^2)} \frac{|A(n_1, n_2)|}{n_2^{1/3}} \ll n_1 \left( \frac{\sqrt{N} T^\varepsilon}{M^3 n_1^2} \right)^{2/3}$$

and easily

$$\sum_{n_1 | cm} \frac{1}{n_1} \ll (cm)^\varepsilon.$$

Plugging these in we see

$$\tilde{\mathcal{R}}_2^- \ll T^{2+5\varepsilon/3} M^{-1} N^{-1/2} \sum_{m \leq \sqrt{2N}} m^{1+\varepsilon} \sum_{c \asymp \frac{\sqrt{N}}{Tm}} c^{1+2\varepsilon}.$$

Now

$$\sum_{c \asymp \frac{\sqrt{N}}{Tm}} c^{1+2\varepsilon} \ll \left( \frac{\sqrt{N}}{Tm} \right)^{2+2\varepsilon} \text{ and } \sum_{m \leq \sqrt{2N}} \frac{1}{m^{1+\varepsilon}} \ll \frac{1}{\varepsilon}.$$

Consequently,

$$\tilde{\mathcal{R}}_2^- \ll T^{\frac{3}{2} + \frac{13}{6}\varepsilon} M^{-1}.$$

This is clearly smaller than  $T^{1+\varepsilon} M$  by a power of  $T$  if  $M \geq T^{\frac{1}{3}+2\varepsilon}$ .  $\square$

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